CALCULUS AND QUANTIZATIONS OVER HOPF ALGEBRAS

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Abstract. In this paper we outline an approach to calculus over quasitriangular Hopf algebras. We construct braided differential operators and introduce a general notion of quantizations in monoidal categories . We discuss some applications to quantizations of differential operators.

0.Introduction

In this paper we study differential operators in the framework of monoidal categories equipped with a braiding or symmetry. To be more concrete, we choose as an example the category of modules over quasitriangular Hopf algebra.

We introduce (braided) differential operators in a purely algebraic manner. This gives us a possibility to develop calculus in an intrinsic way without enforcing any type of Leibniz rule.

A general notion of quantization in monoidal categories, proposed in this paper, is a natural isomorphism of the tensor product bifunctor equipped with some natural coherence conditions. The quantization "deforms" all natural algebraic and differential objects in the monoidal category.

There are now a number of different approaches to the construction of a calculus: the universal construction for associative algebras [C],[K],[DV], fermionic and colour calculus [JK],[BMO],[KK], the calculus for quadratic algebras [WZ],[M], covariant calculus on Hopf algebras [W], etc. Here we would like to illustrate the general scheme for differential calculus suggested in [L1],[L2] on the example of the monoidal category of modules over quasitriangular Hopf algebra.

The paper is organized as follows. In section 1 we build up modules of (braided) differential operators in the category of modules over quasitriagular Hopf algebra. In section 2 we consider (braided) derivations as a special type of 1-st order (braided) differential operators. We show that these operators may be described by a braided Leibniz rule. We should note that for general braidings (not symmetries) the rule consists of four identities.

As usual we introduce braided differential 1-forms $\Omega^1(A)$ as a representative object for the functor of braided derivations in new category with morphisms generated by braided differential operators of degree 0.

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Our construction of braided differential forms and the de Rham complex is based on the following two assumptions:

(1) An algebra of braided differential forms should be a braided commutative algebra generated by the base algebra A and symmetric bimodule $\Omega^1(A)$,and
(2) de Rham differential d should be a braided derivation of the algebra, such that $d^2 = 0$.

To analyze the first condition we describe all braidings in the category of $\mathbb{Z}$–graded objects over a given monoidal category and show that the second condition defines a special class of braidings which we call differential prolongations of the given one.

It is an experimental fact that modules of braided derivations (in various definitions) have no good Lie algebra structure. We remark that modules of braided differential forms may be considered as Lie coalgebras and condition $d^2 = 0$ may be considered as the analogue of co- Jacobi identity.

In section 3 we introduce a quantization of functors acting between monoidal categories. In our definition, a quantization is a natural isomorphism equipped with some natural coherence conditions and the functor considered together with a quantization is simply a monoidal functor [cf.McL, Ep].

We suggest two ways for calculation of quantizations. One of them reduces the calculation to nonlinear cohomologies. The other describes quantizations in terms of multiplicative Hochschild cohomologies of the Grothendieck ring of the given monoidal category. These constructions are illustrated by some examples. Thus for the monoidal category of representations of torus the quantizers may be described in terms of invariant Poisson structures. Their construction given in 3.6. produces the Moyal quantizations[BFFLS,V]. In the same way we obtain series of quantizers for categories of representations of compact Lie groups.

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1.Braided differential operators over quasitriangular Hopf algebras

1.1. Let k be a commutative ring with unit. We shall assume that all k–algebras under consideration have a unit, and that all algebra homomorphisms are unit–preserving.

Let H be a Hopf k–algebra with a coproduct $\Delta : H \longrightarrow H \otimes H$, counit $\varepsilon : H \longrightarrow k$ and antipode $S : H \longrightarrow H$.

Denote by $\mathcal{C} = \mathcal{M}od_H$ the category of left H–modules. Morphisms in this category are H–module homomorphisms.

To convert $\mathcal{C}$ into a monoidal category, we define a bifunctor of tensor product

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

to be the usual tensor product of modules over $k : X \otimes Y = X \underset{k}{\otimes} Y$.

We define an H–module structure in the tensor product as follows:

$$h(x \otimes y) = \sum_h h_{(1)}(x) \otimes h_{(2)}(y),$$

where $x \in X, y \in Y, h \in H$ and $\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)}$ in the Sweedler notations [S].

The bifunctor of internal homomorphisms $X, Y \mapsto Hom(X, Y)$ in the monoidal category is the adjoint bifunctor for the tensor product bifunctor.

In our case $Hom(X, Y)$ coincides with the module of k–morphisms $Hom_k(X, Y)$ equipped with the following H–module structure :

$$h(f)(x) = \sum_h h_{(1)} \cdot f(S(h_{(2)})(x)),$$

where $h \in H, x \in X, f \in Hom(X, Y)$.

1.2. By the usual definition of algebras in monoidal categories algebras in the category $\mathcal{C}$ are H–module algebras, i.e. k–algebras A which are: (1) H–modules, (2) multiplication maps $\mu : A \otimes A \longrightarrow A, \quad \mu(a \otimes b) = a \cdot b$, are morphisms in $\mathcal{C}$.

The last condition means that

$$h(a \cdot b) = \sum_h h_{(1)}(a) \cdot h_{(2)}(b),$$

and

$$h(1) = \varepsilon(h),$$

for all $a, b \in A, \quad h \in H$.

A left A–module P in the category $\mathcal{C}$ is an H– and A– module

$$\mu^l : A \otimes P \longrightarrow P, \quad \mu^l(a \otimes p) = a \cdot p,$$

such that

$$h(a \cdot p) = \sum_h (h_{(1)}(a)) \cdot (h_{(2)}(p))$$

for all $a \in A, \quad p \in P$.

In the same way one defines right A–modules and $A-A$ bimodules in the category $\mathcal{C}$.

1.3. Recall [JS] that a *braiding* in a monoidal category $\mathcal{C}$ is a natural isomorphism

$$\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X, \qquad \forall X, Y \in \mathcal{O}b(\mathcal{C}),$$

such that the following hexagon conditions hold:

$$\begin{cases} \sigma_{X \otimes Y, Z} = \big(\sigma_{X,Z} \otimes id_Y\big) \circ \big(id_X \otimes \sigma_{Y,Z}\big), \\ \sigma_{X, Y \otimes Z} = \big(id_Y \otimes \sigma_{X,Z}\big) \circ \big(\sigma_{X,Y} \otimes id_Z\big). \end{cases}$$

It is easy to show that any braiding in the category is given by the *braiding element* $\sigma \in H \otimes H$, such that the above hexagon conditions

$$\begin{cases} \big(id_H \otimes \Delta\big)(\sigma) = (\sigma \otimes 1) \cdot \sigma_{13}, \\ \big(\Delta \otimes id_H\big)(\sigma) = (1 \otimes \sigma) \cdot \sigma_{13}, \end{cases} \tag{1}$$

and the condition that σ is a $\mathcal{C}$–morphism

$$\sigma \cdot \tau((\Delta(h)) = \Delta(h) \cdot \sigma \tag{2}$$

hold for all $h \in H$.

Here we denote by

$$\tau : H \otimes H \longrightarrow H \otimes H$$

the natural *twist* $\tau(a \otimes b) = b \otimes a$.

As usual, $\sigma_{1,3}$ denoted the element of $H \otimes H \otimes H$ which is σ in the 1-st and 3-rd factors, i.e. $\sigma_{13} = \sum \sigma' \otimes 1 \otimes \sigma''$, if $\sigma = \sum \sigma' \otimes \sigma''$, or $\sigma_{13} = (id_H \otimes \tau)(\sigma \otimes 1)$.

The definition of the braiding by means of the braiding element is the following:

$$\sigma_{X,Y}(x \otimes y) = \sigma \cdot (y \otimes x) = \sum \sigma'(y) \otimes \sigma''(x),$$

for all $x \in X, y \in Y$.

Following Drinfeld [D] a Hopf algebra H equipped with a braiding element σ is called a *quasitriangular Hopf algebra.*

In a quasitriangular Hopf algebra one has the following relations on the braiding element σ:

$$\sigma_{12}\sigma_{13}\sigma_{23} = \sigma_{23}\sigma_{13}\sigma_{12}, \quad \text{(Quantum Yang–Baxter equation)}$$

where $\sigma_{12} = \sigma \otimes 1, \sigma_{23} = 1 \otimes \sigma,$ and

$$\begin{cases} (S \otimes id)(\sigma) = (id \otimes S)(\sigma) = \sigma^{-1}, & (S \otimes S)(\sigma) = \sigma, \\ (\varepsilon \otimes id)(\sigma) = (id \otimes \varepsilon)(\sigma) = 1. \end{cases}$$

A braiding σ is called a *symmetry* [McL] if $\sigma_{Y,X} \circ \sigma_{X,Y} = id_{X,Y}$, or $\sigma \cdot \tau(\sigma) = 1$ in terms of braiding elements.

A quasitriangular Hopf algebra (H, σ) is called a *triangular Hopf algebra* if σ is a symmetry.

Examples.

(1) Let G be a finite multiplicative group and $H = k(G)$ be a k–algebra of functions on G with values in k.

Define a coproduct, a counit an and antipode as usual:

$$(\Delta f)(x, y) = f(xy), \quad \varepsilon(f) = f(e), \quad S(f)(x) = f(x^{-1}),$$

where $f \in k(G)$, $x, y \in G$, e is a unit of G and we identify $k(G) \otimes k(G)$ with $k(G \times G)$.

Let θ_x, $x \in G$ be a basis in $k(G)$ such that:$\theta_x(x) = 1$, and $\theta_x(y) = 0$, if $x \neq y$.

In terms of this basis we have:

(1) $\theta_x \cdot \theta_x = \theta_x$, and $\theta_x \cdot \theta_y = 0$, if $x \neq y$,
(2) $\Delta\theta_x = \sum_{y \in G} \theta_y \otimes \theta_{y^{-1}x}$,
(3) $\varepsilon(\theta_x) = 0$, if $x \neq e$, and $\varepsilon(\theta_e) = 1$,
(4) $S(\theta_x) = \theta_{x^{-1}}$.

Property (1) shows that θ_x are projectors and therefore the category Mod_H coincides with a category of G–graded modules, $X = \sum_{g\in G} X_g$, equipped with the following H–action:

$$\theta_g(\sum_{h\in G} x_h) = x_g.$$

Morphisms $f : X \longrightarrow Y$ in the category are G–graded k–morphisms: $f(X_g) \subset Y_g$.

Tensor product $X \otimes Y$ in this category is the usual G–graded product:

$$(X \otimes Y)_g = \sum_{h\in G} X_h \otimes Y_{h^{-1}g}.$$

Internal homomorphisms $Hom(X, Y)$ coinside with modules k–homomorphisms equipped with the grading:

$$Hom(X, Y)_g = \{f \in Hom_k(X, Y)|f(X_h) \subset Y_{gh}, \forall h \in G\}.$$

A braiding element $\sigma \in H \otimes H$ can be written down as follows:

$$\sigma = \sum_{a,b\in G} \sigma(a, b)\, \theta_a \otimes \theta_b. \tag{1}$$

From the hexagon axioms we obtain the following conditions on the function $\sigma(a, b)$:

$$\sigma(ab, c) = \sigma(a, c) \cdot \sigma(b, c), \tag{2}$$

and

$$\sigma(a, bc) = \sigma(a, b) \cdot \sigma(a, c), \tag{3}$$

for all $a, b, c \in G$.

But condition 1.3.(2) holds if and only if G is an Abelian group.

Therefore, any braiding in the category of G–graded modules over an Abelian group G is given by the group bihomomorphism

$$\sigma : G \times G \longrightarrow U(k),$$

where $U(k)$ is the unit group of the ring k.

The braiding is a symmetry if and only if the following multiplicative skew symmetry property holds: $\sigma(a, b)\sigma(b, a) = 1$.

(2) Let G be a finite group and $H = k[G]$ the group algebra, $k[G] = (k(G))^*$.

Denote by δ_g, $g \in G$ the dual basis of δ–functions: $\delta_g(\theta_g) = 1$, and $\delta_g(\theta_h) = 0$, if $g \neq h$.

In terms of this basis a Hopf algebra structure has the following form:

$$\delta_g \cdot \delta_h = \delta_{gh}, \quad \Delta(\delta_g) = \delta_g \otimes \delta_g, \quad S(\delta_g) = \delta_{g^{-1}}, \quad \varepsilon(\delta_g) = 1, \delta_e = 1.$$

The category Mod_H is a category Mod_G of left G–modules over k.

A braiding element $\sigma \in H \otimes H$, $\sigma = \sum_{a,b \in G} \sigma(a,b)\,\delta_a \otimes \delta_b$, in the case $k = \mathbb{C}$, may be considered as a $\mathbb{C}$–morphism

$$\hat{\sigma} : \mathbb{C}(G) \longrightarrow \mathbb{C}[G],$$

where $\hat{\sigma}(\theta_a) = \sum_{b \in G} \sigma(a,b)\,\delta_b$.

There is the following description [L4] of braided elements. Let us fix two central subgroups $H_1, H_2 \subset \mathcal{Z}(G)$ and a group homomorphism $\phi : \hat{H}_2 \longrightarrow H_1$, where $\hat{H}_2$ is a dual group for H_2.

Then any braiding in the category can be obtained from the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}(G) & \xrightarrow{\hat{\sigma}} & \mathbb{C}[G] \\ {\scriptstyle r_1}\downarrow & & \uparrow{\scriptstyle (r_2)^*} \\ \mathbb{C}(H_1) & \xrightarrow{\tilde{\phi}} & [H_2] \end{array}$$

where $r_1 : \mathbb{C}(G) \longrightarrow \mathbb{C}(H_1)$ is the restriction map, and $(r_2)^* : \mathbb{C}[H_2] \longrightarrow \mathbb{C}[G]$ is the adjoint of r_2, and $\tilde{\phi}$ is given by the composition

$$\tilde{\phi} : \mathbb{C}(H_1) \xrightarrow{\phi^*} \mathbb{C}(\hat{H}_2) \xrightarrow{\mathcal{F}} \mathbb{C}[H_2],$$

where $\mathcal{F}$ is the Fourier transform.

(3) The same description of the braiding elements holds for any compact Lie group G, [L4].

(4) Let G be a finite group. To introduce a braiding in the category of G– graded modules we need some additional structures.

Consider for example G–graded modules $X = \sum_{g \in G} X_g$ equipped with G–action, such that $h(X_g) \subset X_{ghg^{-1}}$.

In this case our Hopf algebra H is a smash product $k(G)\#k[G]$, i.e. a Hopf algebra generated by products $\theta_g\,\delta_h$ with the following relations: $\delta_h\,\theta_g = \theta_{hgh^{-1}}\,\delta_h$.

Define a braiding in this category as follows:

$$\sigma_{X,Y}(x_g \otimes y_h) = g(y_h) \otimes x_g, \tag{1}$$

where $X = \sum_{g \in G} X_g, \quad Y = \sum_{h \in G} Y_h$, and $x_g \in X_g, y_h \in Y_h$.

It is easy to check that σ is a braiding.

One can deform the braiding by some function $s : G \times G \longrightarrow U(k)$:

$$\sigma_{X,Y}(x_g \otimes y_h) = s(g,h)\,g(y_h) \otimes x_g. \tag{2}$$

Then σ is a braiding in the category if the following conditions hold

$$s(aga^{-1}, aha^{-1}) = s(g,h), \tag{3}$$

$$s(f, hg) = s(f,h)\,s(f,g), \tag{4}$$

and

$$s(fh, g) = s(f, hgh^{-1})s(h,g), \tag{5}$$

for all $a, f, gh \in G$.

We say that a normalized function s, $s(e,g) = s(g,e) = 1$, is a *colour* on the group if conditions (3,4,5) hold.

Note that as a rule braidings (1) and (2) are not symmetries.

The braiding element σ in this case has the following form:

$$\sigma = \sum_{g,h\in G} s(g,h)\delta_g\theta_h \otimes \theta_g.$$

1.4. Let A be an algebra in the category and let X be $A - A$ bimodule in the category.

Denote by $\mu : A\otimes A \longrightarrow A$, $\mu(a\otimes b) = a\cdot b$, a multiplication in the algebra and by $\mu^r : X\otimes A \longrightarrow X$, $\mu^r(x\otimes a) = xa$ the right multiplication and by $\mu^l : A\otimes X \longrightarrow X$, $\mu^l(a\otimes x) = ax$, the left multiplication in the bimodule.

By using a braiding σ we introduce new multiplications:

$$\mu_\sigma = \mu\circ\sigma_{A,A}, \quad \mu^l_\sigma = \mu^r\circ\sigma_{A,X}, \quad \mu^r_\sigma = \mu^l\circ\sigma_{X,A},$$

and denote the new multiplications by $*$:

$$a*b = \mu_\sigma(a\otimes b), \quad a*x = \mu^l_\sigma(a\otimes x), \quad x*a = \mu^r_\sigma(x\otimes a),$$

where $a, b\in A, x\in X$.

In terms of the braiding element σ we have

$$a*b = \sum\sigma'(b)\sigma''(a), \quad a*x = \sum\sigma'(x)\sigma''(a), \quad x*a = \sum\sigma'(a)\sigma''(x).$$

Proposition.

(1) *The pair (A,μ_σ) determines an algebra structure in the category.*
(2) *The triple $(X,\mu^l_\sigma,\mu^r_\sigma)$ determines $(A,\mu_\sigma)-(A,\mu_\sigma)$ bimodule structure in the category.*

Proof. We show, for example, that μ^l_σ determines a left (A,μ_σ)–structure and that the multiplications μ^r_σ and μ^l_σ commute.

For the first one we have

$$\begin{aligned} a*(b*x) &= \sum\sigma'(b*x)\,\sigma''(a) = \sum\sigma'_{(1)}\tilde\sigma'(x)\,\sigma_{(2)}\tilde\sigma''(b)\,\sigma''(a) \\ &\overset{\text{hexagon}}{=} \mu^r\circ(id\otimes\mu)(\sigma_{23}\sigma_{13}\sigma_{12})(x\otimes b\otimes a) \end{aligned}$$

and

$$\begin{aligned} (a*b)*x &= \sum\sigma'(x)\,\sigma''(a*b) = \sum\sigma'(x)\,\sigma''_{(1)}\tilde\sigma'(b)\,\sigma''_{(2)}\tilde\sigma''(a) \\ &\overset{\text{hexagon}}{=} \mu^r(id\otimes\mu)(\sigma_{12}\sigma_{13}\sigma_{23})(x\otimes b\otimes a). \end{aligned}$$

Hence, the Yang–Baxter equation implies the left (A,μ_σ)– module structure.

Comparing terms $(a*x)*b$ and $a*(x*b)$ we get

$$(a*x)*b = \sum \sigma'(b)\,\sigma''(a*x) = \sum \sigma'(b)\,\sigma''_{(1)}\tilde{\sigma}'(x)\,\sigma''_{(2)}\tilde{\sigma}''(a)$$
$$\overset{\text{hexagon}}{=} \mu^l \circ (id \otimes \mu^r)(\sigma_{12}\sigma_{13}\sigma_{23})(b \otimes x \otimes a),$$

and

$$a*(x*b) = \sum \sigma'(x*b)\,\sigma''(a) = \sum \sigma'_{(1)}\tilde{\sigma}'(b)\,\sigma'_{(2)}\tilde{\sigma}''(x)\,\sigma''(a)$$
$$\overset{\text{hexagon}}{=} \mu^l \circ (id \otimes \mu^r)(\sigma_{23}\sigma_{13}\sigma_{12}(b \otimes x \otimes a)).$$

Therefore, $(A,\mu_\sigma)-(A,\mu_\sigma)$ bimodule structure follows from Yang–Baxter equation too. $\square$

An algebra A in the category $\mathcal{C}$ is called a *σ–commutative algebra* if $\mu_\sigma = \mu$, or equivalently, if

$$a \cdot b = a * b = \sum \sigma'(b) \cdot \sigma''(a),$$

for all $a, b \in A$.

An $A-A$ bimodule X in the category $\mathcal{C}$ is called *σ–symmetric* if $\mu^r_\sigma = \mu^r$ and $\mu^l_\sigma = \mu^l$, or in terms of braiding element σ if

$$x \cdot a = x * a = \sum \sigma'(a) \cdot \sigma''(x), \quad a \cdot x = a * x = \sum \sigma'(x) \cdot \sigma''(a),$$

for all $a \in A, x \in X$.

Define a *σ–symmetric part* X_σ of any $A-A$ bimodule X in the category as follows:

$$X_\sigma = \{x \in X \mid a \cdot x = a * x, \quad x \cdot a = x * a, \quad \forall a \in A\}.$$

Theorem. *Let A be a σ–commutative algebra in the category $\mathcal{C}$ and let X be an $A-A$ bimodule. Then X_σ is a σ–symmetric $A-A$ bimodule.*

Proof. It is enough to show that $ax \in X_\sigma$, and $xa \in X$, if $x \in X_\sigma, a \in A$.

Let us prove, for example, the first inclusion.

For any elements $a, b \in A, x \in X_\sigma$ one has

$$b*(ax) = b*(a*x) = (b*a)*x = (ba)*x = (ba)x = b(ax).$$

Hence, $ax \in X_\sigma$. $\square$

Examples.

(1) Let G be an Abelian group. Consider the category of G–graded modules with a braiding σ is given by the group bihomomorphism $\sigma : G \times G \longrightarrow U(k)$, (see ex.1.3.(1)).

An algebra in this category is a G–graded algebra $A = \sum_{g \in G} A_g, \quad A_g\, A_h \subset A_{gh}$. The algebra is a σ–commutative if and only if the following relations hold

$$a_g\, a_h = \sigma(g,h)\, a_h\, a_g,$$

for all $a_g \in A_g, \quad a_h \in A_h$.

(2) One can reformulate the FRT-construction [FRT] of function algebras on quantum groups in the case of G–graded modules in the following way.

Let $X = \sum_{g\in G} X_g$ be a G–graded module and $X^* = \sum_{g\in g} X_g^*$ be the dual:$X_g^* = (X_{g^{-1}})^*$. Consider a new module $Y = X\otimes X^*$. This module is generated by elements $y_{g,h} = x_g \otimes y_h^*$, of degree gh^{-1} and the braiding takes the form:

$$\sigma_{X,Y} : y_{a,b} \otimes y_{c,d} \mapsto \sigma(ab^{-1}, cd^{-1})\, y_{c,d} \otimes y_{a,b}.$$

The σ–symmetric algebra generated by elements of $y_{a,b}$ is a factor of the tensor algebra $T(Y)$ by two-sided ideal generated by $Im(\sigma - 1)$.

Note that this algebra is not trivial for any braiding σ.

(3) For the case of the category of G-graded G–modules (see ex.1.3.(3)) an algebra in the category is a G–graded algebra $A = \sum_{g\in G} A_g$ equipped with G–action: $g(A_h) \subset A_{ghg^{-1}}$.

The condition of σ–commutativity for the given colour s takes the form:

$$a_g \cdot a_h = s(g,h)\, g(a_h) \cdot a_g.$$

(4) Consider the group algebra $k[G]$ equipped with a G action:

$$g(\delta_h) = \delta_{ghg^{-1}},$$

as an algebra in the category of G–graded G–modules.

The algebra will be σ–commutative with respest to braiding (1):

$$\delta_g \cdot \delta_h = g(\delta_h) \cdot \delta_g.$$

(5) Crossed products.

Let $\omega : G \times G \longrightarrow U(k)$ be a multiplicative G–invariant $(\omega(aga^{-1}, aha^{-1}) = \omega(g,h), \quad \forall a,g,h \in G)$ 2-cocycle on the group.

The crossed product $k_\omega[G]$ coincides with $k[G]$ as a G–graded G– module but has a new multiplication:

$$\delta_g * \delta_h = \omega(g,h)\delta_{gh}.$$

In this case the function

$$s(g,h) = \omega(g,h) \cdot \omega(h,g)^{-1}$$

determines a colour on the group, and $k_\omega[G]$ is a σ–commutative algebra.

(6) Let $G = \mathbb{Z}^n$, $k = \mathbb{C}$, and let $\Theta = \|\theta_{ij}\| \in Mat_n(\mathbb{R})$ be a matrix such that $\theta_{ij} = 0$, if $i \le j$. Taking the twisting 2-cocycle ω in the form

$$\omega(x,y) = exp(\pi\imath\langle\Theta x, y\rangle),$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{Z}^n$, and $\langle x, y\rangle = \sum x_i y_i$, we get a σ–commutative algebra $\mathbb{C}_\omega[\mathbb{Z}^n]$, which is called a quantum torus.

1.5. Here we apply the above procedure to modules of internal homomorphisms and left modules.

Let P and Q be left A–modules. Then the module of internal homomorphisms $Hom(P,Q)$ is endowed with an $A-A$ bimodule structure with respect to left and right multiplication

$$l_a(f)(p) = af(p), \quad r_a(f)(p) = f(ap).$$

Denote by $Hom_\sigma(P,Q) \subset Hom(P,Q)$ the σ–symmetric part of the bimodule.

Elements of $Hom_\sigma(P,Q)$ are called *σ–homomorphisms.*

Therefore, an internal homomorphism $f : P \longrightarrow Q$ is a σ–homomorphism if and only if the following relations hold:

$$f(a \cdot p) = \sum \sigma'(a) \cdot \sigma''(f)(p), \tag{1}$$

and

$$a \cdot f(p) = \sum \sigma'(f)(\sigma''(a) \cdot p), \tag{2}$$

for all $a \in A, p \in P$.

Remark that if $f : P \longrightarrow Q$ is simultaneously a morphism in the category and a σ–homomorphism, then $h(f) = \varepsilon(h)\, f$, for all $h \in H$, and therefore the above conditions are reduced to the following known one: $f(a\, p) = a\, f(p)$.

Theorem. *Let A be a σ–commutative algebra. Then for any left A–module P with left multiplication $\mu^l : A \otimes P \longrightarrow P$, right multiplication $\mu^r = \mu^l \circ \sigma_{P,A} : P \otimes A \longrightarrow P$ determines $A-A$ bimodule structure.*

Proof. We should show that the right and the left structures commute.

One has

$$(ap)b = \sum \sigma'(b)\sigma''(ap) = \sum \sigma'(b)\sigma''_{(1)}(a)\sigma''_{(2)}(p) =$$
$$\sum \sigma'\tilde{\sigma}'(b)\sigma''(a)\tilde{\sigma}''(p) = \sum a\tilde{\sigma}'(b)\tilde{\sigma}''(p) = a(bp)$$

□

Let P be a left A–module over a σ–commutative algebra A. Consider P as $A-A$ bimodule. We introduce a bimodule P_σ which is the σ– symmetric part of the bimodule P.

We have the following direct description of the bimodule:

$$P_\sigma = \left\{ p \in P \mid (\mu^l - \mu^l \circ \sigma_{P,A} \circ \sigma_{A,P})(a \otimes p) = 0, \quad \forall a \in A \right\}.$$

In terms of the braiding element we get

$$p \in P_\sigma \Leftrightarrow ap = \sum \sigma'(p) \cdot \sigma''(a) \overset{\mathrm{def}\mu^r}{=} \sum \tilde{\sigma}'\sigma''(a) \cdot \tilde{\sigma}''\sigma'(p),$$

or, if we set $\gamma = \sigma \cdot \tau(\sigma) = \sum \gamma' \otimes \gamma''$, where $\gamma' = \sum \tilde{\sigma}' \cdot \sigma''$, and $\gamma'' = \sum \tilde{\sigma}'' \cdot \sigma'$, then

$$P_\sigma = \left\{ p \in P \mid a \cdot p = \sum \gamma'(a) \cdot \gamma''(p) \right\}.$$

The A-submodule $P_\sigma \subset P$ will be called the σ– *symmetric part* of the left A–module P.

A left A–module P is called *σ–symmetric* if $P_\sigma = P$.

1.6. The following theorem describes the relation between two natural σ–symmetric bimodules associated to a given left module.

Theorem. *Let A be a σ–commutative algebra and let P be a left A–module in the category $\mathcal{C}$. Then there is an isomorphism between $Hom_\sigma(A,P)$ and P_σ is given by the formula*

$$f \in Hom_\sigma(A,P) \mapsto f(1) \in P_\sigma.$$

Proof. Let $f \in Hom_\sigma(A,P)$ then for any elements $a, b \in A$ we have

$$f(ab) = \sum \sigma'(a) \cdot \sigma''(f)(b).$$

Hence, if we set $b = 1, \quad p = f(1)$, we get

$$\begin{aligned} f(a) &= \sum \sigma'(a) \cdot \sigma''(f)(1) = \sum \sigma' \sigma''_{(1)}(f(S(\sigma_{(2)})(1))) \\ &= \sum \sigma'(a) \cdot \sigma''_{(1)}(f(\varepsilon S(\sigma_{(2)})) = \sum \sigma'(a) \cdot (\sigma''_{(1)} \varepsilon(\sigma_{(2)}))p \\ &= \sum \sigma'(a) \cdot \sigma''(p). \end{aligned}$$

From condition 1.5.(2) one gets

$$a \cdot f(b) = \sum \sigma'(f)(\sigma''(a)b).$$

Hence,

$$\begin{aligned} &a \cdot p = \sum \sigma_{(1)} f(S(\sigma'_{(2)})\sigma''(a)) = \sum \sigma_{(1)}(\tilde{\sigma}' S(\sigma'_{(2)})\sigma''(a) \cdot \tilde{\sigma}''(p) = \\ &\sum (\sigma'_{(1)} \tilde{\sigma}' S(\sigma'_{(3)})\sigma''(a)) \cdot (\sigma'_{(2)} \tilde{\sigma}''(p)) \overset{1.3.(3)}{=} \sum (\tilde{\sigma}' \sigma'_{(2)} S(\sigma'_{(3)})\sigma''(a) \cdot (\tilde{\sigma}'' \sigma'_{(1)}(p)) = \\ &\sum (\tilde{\sigma}' \varepsilon(\sigma'_{(2)})\sigma''(a)) \cdot (\tilde{\sigma}'' \sigma'_{(1)}(p)) = \sum (\tilde{\sigma}' \sigma''(a)) \cdot (\tilde{\sigma}'' \sigma'(p)), \end{aligned}$$

where $\sigma = \sum \sigma' \otimes \sigma'' = \sum \tilde{\sigma}' \otimes \tilde{\sigma}''$.

Therefore, $p \in P_\sigma$.

Similar calculations show that for any element $p \in P_\sigma$ the formula

$$f_p(a) = \sum \sigma'(a) \cdot \sigma''(p) \tag{1}$$

determines a σ–homomorphism $f_p \in Hom_\sigma(A,P)$. $\square$

1.7. Let A be a σ–commutative algebra and X an $A - A$ bimodule.

Consider a quotient bimodule X/X_σ and define a bimodule $X_\sigma^{(1)} \subset X$ as the inverse image of the bimodule $(X/X_\sigma)_\sigma \subset X/X_\sigma$ with respect to the natural projection $X \longrightarrow X/X_\sigma$.

Thus we get an embedding $X_\sigma \subset X_\sigma^{(1)}$ and $X_\sigma^{(1)}/X_\sigma$ is a σ–symmetric bimodule by construction.

Proceeding in this way, we obtain a filtration of the bimodule X by bimodules $X_\sigma^{(i)}, \quad i = -1, 0, 1, \ldots$:

$$0 = X_\sigma^{(-1)} \subset X_\sigma^{(0)} = X_\sigma \subset X_\sigma^{(1)} \subset \cdots \subset X_\sigma^{(i)} \subset X_\sigma^{(i+1)} \subset \cdots \subset X_\sigma^{(*)} \subset X$$

where, by definition, $X_\sigma^{(i+1)} \subset X$ is the inverse image of $(X/X_\sigma^{(i)})_\sigma$ with respect to the projection $X \longrightarrow X/X_\sigma^{(i)}$.

Note that all the quotients $X_\sigma^{(i)}/X_\sigma^{(i-1)}$ are σ–symmetric modules by the construction.

We call the bimodule $X_\sigma^{(*)} = \bigcup X_\sigma^{(i)}$ a *differential approximation* of the $A - A$ bimodule X.

To produce more concrete description of the differential approximation, we define two types of morphisms:

$$\delta_a^l(x) = a \cdot x - a * x,$$

and

$$\delta_a^r(x) = x \cdot a - x * a.$$

Then, by definition, we have

$$X_\sigma = X_\sigma^{(0)} = \left\{ x \in X \;\middle|\; \delta_a^l(x) = \delta_a^r(x) = 0, \forall a \in A \right\},$$

and

$$X_\sigma^{(i)} = \left\{ x \in X \;\middle|\; \delta_a^l(x) \in X_\sigma^{(i-1)}, \delta_\alpha^r(x) \in X_\sigma^{(i-1)}, \forall a \in A \right\}.$$

Examples.

(1) In the category of G–graded modules over commutative group G, an $A - A$ bimodule $X = \sum_{g \in G} X_g$ is a G–graded bimodule such that

$$A_h \cdot X_g \subset X_{hg}, \quad X_g \cdot A_h \subset X_{gh}.$$

The bimodule is σ–symmetric if and only if

$$a_g \cdot x_h = \sigma(g,h)\, x_h \cdot a_g, \quad \text{and} \quad x_h \cdot a_g = \sigma(h,g)\, a_g \cdot x_h,$$

for all $x_h \in X_h, \quad a_g \in A_g, \quad g, h \in G.$

The σ–symmetric part of any $A - A$ bimodule X is $X_\sigma = \sum_{g \in G} (X_\sigma)_g$, where

$$(X_\sigma)_g = \{ x_g \in X_g | a_h x_g = \sigma(h,g)\, x_g a_h, x_g a_h = \sigma(g,h)\, a_h x_g, \forall a_h \in A_h, h \in G \}.$$

The δ–morphisms have the following form

$$\begin{aligned} \delta_{a_h}^l(x_g) &= a_h \cdot x_g - \sigma(h,g)\, x_g \cdot a_h, \\ \delta_{a_h}^r(x_g) &= x_g \cdot a_h - \sigma(g,h)\, a_h \cdot x_g. \end{aligned}$$

Then

$$(X_\sigma)_g = \{ x_g \in X_g | \delta_{a_h}^l(x_g) = \delta_{a_h}^r(x_g) = 0, \quad \forall a_h \in A_h, h \in G \},$$

and $X_\sigma^{(i)} = \sum_{g \in G} (X_\sigma^{(i)})_g$, where

$$(X_\sigma^{(i)})_g = \{ x_g \in X_g | \delta_{a_h}^l(x_g) \in (X_\sigma^{(i-1)})_{(hg)}, \delta_{a_h}^r(x_g) \in (X_\sigma^{(i-1)})_{(gh)}, \forall a_h \in A_h \}.$$

(2) In the category of G–graded G–modules (see ex.1.3.(4)) with braiding σ given by colour s we have, accordingly,

$$\delta^l_{a_h}(x_g) = a_h \cdot x_g - s(h,g)h(x_g) \cdot a_h,$$
$$\delta^r_{a_h}(x_g) = x_g \cdot a_h - s(g,h)g(a_h) \cdot x_g,$$

and the same description of $X^{(i)}_\sigma$.

1.8. Applying the above procedure to bimodules of internal homomorphisms $X = Hom(P,Q)$, we obtain modules of *braided differential operators*:

$$Diff^\sigma_i(P,Q) = (Hom(P,Q))^{(i)}_\sigma$$

in the category $\mathcal{C}$.

Keeping in the mind the definition of differential approximations, we build up the modules of braided differential operators over quasitriangular Hopf algebra in a direct way.

To do this, we introduce two types of morphisms in $Hom(P,Q)$:

$$\delta^l_a(f)(p) = a \cdot f(p) - \sum \sigma'(f)(\sigma''(a) \cdot p),$$

and

$$\delta^r_a(f)(p) = f(a \cdot p) - \sum \sigma'(a) \cdot \sigma''(f)(p),$$

for all $a \in A, p \in P, f \in Hom(P,Q)$.

Then

$$Diff^\sigma_0(P,Q) = Hom_\sigma(P,Q) = \{f \in Hom(P,Q) \mid \delta^r_a(f) = \delta^l_a(f) = 0, \forall a \in A\},$$

and

$$Diff^\sigma_i(P,Q) = \{f \in Hom(P,Q) \mid \delta^r_a(f), \delta^l_a(f) \in Diff^\sigma_{(i-1)}(P,Q), \forall a \in A\}.$$

Examples.

(1) In the category of G–graded modules we have

$$\delta^l_{a_h}(f)(x) = a_h\, f(x) - \sigma(h,g)\, f(a_h x),$$
$$\delta^r_{a_h}(f)(x) = f(a_h x) - \sigma(g,h)\, a_h f(x),$$

if $f \in Hom(X,Y)$ is a homomorphism of degree g.

In the case $\delta^l_{a_h}(f)$ and $\delta^r_{a_h}(f)$ are homomorphisms of degree hg.

(2) In the category of G–graded G–modules we have

$$\delta^l_{a_h}(f)(x) = a_h\, f(x) - s(h,g)h(f)(a_h x),$$
$$\delta^r_{a_h}(f)(x) = f(a_h x) - s(g,h)g(a_h)\, f(x),$$

where the homomorphism f has degree g and $h(f)(x) = h(f(h^{-1}x))$.

(3) The quantum hyperplane is given by the following data: $k=\mathbb{C}$, $\quad G=\mathbb{Z}^n$, and the twisted 2-cocycle

$$\omega(\bar{a},\bar{b})=q^{\langle\Theta\bar{a},\bar{b}\rangle},$$

where Θ is a skew symmetric $n\times n$ matrix, $q\in\mathbb{C}^*$, $\quad \bar{a},\bar{b}\in\mathbb{Z}^n$.

Let A be a σ–commutative algebra in the category of G–graded modules. Assume that A is generated by elements $x_1,...,x_n$ and the relations

$$x_i\cdot x_j=\omega_{ij}\,x_j\cdot x_i,$$

where ω_{ij} are matrix elements of ω.

Then the algebra of differential operators $Diff_*^\sigma(A,A)$ is a $\mathbb{Z}^n$– graded algebra generated by elements x_i of degree $1_i=(0,...,0,\underset{i}{1},0,...,0)\in\mathbb{Z}^n$ and operators ∂_i of degree -1_i and the following relations

$$\begin{aligned}
&\partial_i\cdot x_j-{\omega_{ij}}^{-1}\,x_j\cdot\partial_i=\delta_{ij},\\
&\partial_i\cdot\partial_j-\omega_{ij}\,\partial_j\cdot\partial_i=0,\\
&x_i\cdot x_j-\omega_{ij}\,x_j\cdot x_i=0.
\end{aligned}$$

1.9. In this section we show that the composition of braided differential operators is a braided differential operator.

We start with the following lemmas.

Lemma 1. *Let A be a σ–commutative algebra and P,Q,R left A–modules. Then for any internal homomorphisms $f\in Hom(Q,R), g\in Hom(P,Q)$ and $h\in H$ we have:*

$$h(f\circ g)=\sum_h h_{(1)}(f)\circ h_{(2)}(g).$$

Proof. From the definition of H–action on the modules of internal homomorphisms we have

$$h(f\circ g)(p)=\sum_h h_{(1)}((f\circ g)(S(h_{(2)})p).$$

On the other side we have

$$\begin{aligned}
&\sum_h h_{(1)}(f)\circ h_{(2)}(g)(p)=\sum_h h_{(1)}(f)(h_{(2)}(g(S(h_{(3)}p)))\\
&=\sum_h h_{(1)}(f(S(h_{(2)}h_{(3)}(g(S(h_{(4)}p))))=\sum_h h_{(1)}\varepsilon(h_{(2)})(f(g(S(h_{(3)}p)))\\
&=\sum_h h_{(1)}((f\circ g)(S(h_{(2)})p)).
\end{aligned}$$

□

Lemma 2. *The following formulae hold*

$$\begin{cases} \delta^r_a(f\circ g) = f\circ\delta^r_a(g) + \sum \delta^r_{\sigma'(a)}(f)\circ\sigma''(g), \\ \delta^l_a(f\circ g) = \delta^l_a(f)\circ g + \sum \sigma'(f)\circ\delta^l_{\sigma''(a)}(g), \end{cases}$$

for all $a\in A, f\in Hom(Q,R), g\in Hom(P,Q)$.

Proof. To prove the first formula it is enough to compare

$$\delta^r_a(f\circ g)(p) = (f\circ g)(ap) - \sum \sigma'(a)(\sigma''_{(1)}(f)\circ\sigma''_{(2)}(g))(p)$$
$$\overset{\text{hexagon}}{=} (f\circ g)(ap) - \sum \sigma'\tilde\sigma'(a)\cdot\sigma''(f)\circ\tilde\sigma''(g)(p),$$

and

$$f\circ\delta^r_a(g)(p) = (f\circ g)(ap) - \sum f(\sigma'(a)\sigma''(g)(p))$$
$$= (f\circ g)(ap) - \sum \delta^r_{\sigma'(a)}(f)(\sigma''(g)(p)) - \sum \tilde\sigma'(\sigma'(a))\tilde\sigma''(f)(\sigma''(g)(p)).$$

Similarly, for the second formula we get

$$\delta^l_a(f\circ g)(p) = a(f\circ g)(p) - \sum \sigma'(f\circ g)(\sigma''(a)p)$$
$$= a(f\circ g)(p) - \sum(\sigma'_{(1)}(f)\circ\sigma'_{(2)}(g))(\sigma''(a)p)$$
$$\overset{\text{hexagon}}{=} a(f\circ g)(p) - \sum(\tilde\sigma'(f)\circ\sigma'(g))(\sigma''\tilde\sigma''(a)p),$$

and

$$a(f\circ g)(p) = (\delta^l_a(f)\circ g)(p) - \sum \sigma'(f)(\sigma''(a)g(p)) =$$
$$(\delta^l_a(f)\circ g)(p) - \sum(\sigma'(f)\circ\delta^l_{\sigma''(a)}(g))(p) + \sum \sigma'(f)\circ\tilde\sigma'(g)(\tilde\sigma''\sigma''(a)(p).$$

□

Lemma. *Maps* $\delta^l : A\otimes Hom(P,Q)\longrightarrow Hom(P,Q)$ *and* $\delta^r : Hom(P,Q)\otimes A\longrightarrow Hom(P,Q)$ *are* H*–morphisms:*

$$h(\delta^l_a(f)) = \sum_h \delta^l_{h_{(1)}(a)}(h_{(2)}(f)),$$

and

$$h(\delta^r_a(f)) = \sum_h \delta^r_{h_{(2)}(a)}(h_{(1)}(f)),$$

for all $a\in A,\quad h\in H,\quad f\in Hom(P,Q)$.

Theorem. *Let* A *be a* σ*–commutative algebra and let* P,Q,R *be left* A*–modules in the category* $\mathcal{C}$.

Then

(1) $f\in Diff^\sigma_i(Q,R), g\in Diff^\sigma_j(P,Q)\Longrightarrow f\circ g\in Diff^\sigma_{i+j}(P,R)$.
(2) $f\in Diff^\sigma_i(A,A), g\in Diff^\sigma_j(A,A)\Longrightarrow [f,g]_\sigma\in Diff^\sigma_{i+j-1}(A,A)$,

where $[f,g]_\sigma = f\circ g - \sum\sigma'(g)\circ\sigma''(f)$ *is a* σ*–commutator of internal homomorphisms.*

Proof. The first part of the theorem is follows from the lemmas. The second part is a consequence of the definition of braided differential operators and part (1) of the theorem. □

2.Braided Calculus

In this chapter we introduce braided derivations as special 1-st order braided differential operators and consider braided differential forms as a representative object for the functor of braided derivations. We construct an algebra of braided differential forms as a new $\hat{\sigma}$–commutative algebra equipped with a universal braided derivation d such that $d^2 = 0$. We show that the main facts of the usual calculus can be translated in the case of arbitrary braidings.

2.1. Let A be a σ–commutative algebra and P be a left A–module in the category $\mathcal{C}$.

We define modules of *braided derivations* as follows

$$D(P) = \{ f \in Diff_1^{\sigma}(A, P) |\, f(1) = 0\}.$$

Elements of $D(P)$ will be called *braided* or *σ–derivations* of the algebra A with values in the module P.

We now produce a description of σ–derivation in terms of generalized (or braided) Leibniz rule.

Let $f \in D(P)$ and $a \in A$. Then $\delta_a^l(f), \delta_a^r(f) \in Hom_{\sigma}(A, P) = P_{\sigma}$ by definition of σ–derivation.

Therefore, by 1.6.(1), we have

$$\begin{cases} \delta_a^r(f)(b) = \sum \sigma'(b) \cdot \sigma''(p_r), \\ \delta_a^l(f)(b) = \sum \sigma'(b) \cdot \sigma''(p_l), \end{cases}$$

for some elements $p_r, p_l \in P_{\sigma}$.

We have

$$\begin{cases} p_l = \delta_a^l(f)(1) = a \cdot f(1) - \sum \sigma'(f)(\sigma''(a)), \\ p_r = \delta_a^r(f)(1) = f(a) - \sum \sigma'(a) \cdot \sigma''(f)(1). \end{cases}$$

Since, $f(1) = 0$, and $h(f)(1) = h_{(1)} \cdot f(S(h_{(2)}(1)) = h_{(1)}\varepsilon(h_{(2)})f(1) = 0$, for all $h \in H$, we get

$$\begin{cases} p_l = -\sum \sigma'(f)(\sigma''(a)), \\ p_r = f(a). \end{cases}$$

Therefore, conditions $p_r \in P_{\sigma}, p_l \in P_{\sigma}$ mean that $f : A \longrightarrow P_{\sigma} \subset P$, and from formulae (1) and (2) we obtain the following form of the *braided Leibniz rule*

$$\begin{cases} f(a \cdot b) = f(a) \cdot b + \sum \sigma'(a) \cdot \sigma''(f)(b), \\ \sum \sigma'(f)(\sigma''(a) \cdot b) = a \cdot f(b) + \sum \sigma'(f)(\sigma''(a)) \cdot b. \end{cases} \tag{1}$$

Summarizing, we obtain the following description of braided derivations.

Proposition. *An internal homomorphism $f : A \longrightarrow P$ is a σ– derivation if and only if*

(1) *$f : A \longrightarrow P_{\sigma} \subset P$, and*
(2) *braided Leibniz rule (1) holds.*

Remark. Let $f : A \longrightarrow P$ be a braided derivation and a morphism in the category. Then $h(f) = \varepsilon(h)\cdot f$, for all $h \in H$, and the braided Leibniz rule takes the usual form:

$$f(ab) = f(a)b + af(b).$$

Examples.

(1) Let A be an algebra in the category of G–graded modules. Then an internal homomorphism $f : A \longrightarrow A$ of degree $g \in G$ is a braided derivation if the following form of the braided Leibniz rules hold:

$$f(a_h\, b) = f(a_h)\, b + \sigma(g,h)a_h\, f(b), \tag{1}$$

and

$$\sigma(h,g)\, f(a_h b) = a_h\, f(b) + \sigma(h,g)\, f(a_h)\, b, \tag{2}$$

for all $a_h \in A_h, \quad b \in A$.

Therefore, in the algebra the folowing relations hold:

$$(\sigma(h,g)\sigma(g,h) - 1)a_h\, f(b) = 0.$$

Remark that we need formula (1) only if σ is a symmetry.

(2) The braided Leibniz rules in the category of G–graded G–modules take the form:

$$f(a_h b) = f(a_h)\, b + s(g,h)g(a_h)\, f(b) \qquad (1,)$$

and

$$s(h,g)h(f)(a_h b) = a_h\, f(b) + s(h,g)h(f)(a_h)\, b, \tag{2}$$

where f as above is an internal homomorphism of degree $g \in G$ and $a_h \in A_h, \quad b \in A$.

Formula (2) implies the following relation

$$a_g\, f(b) = s(h,g)s(g,h)\,(hgh^{-1})(a_h)\, h(f)(b),$$

for all $a_h \in A_h, \quad b \in A$.

(3)Consider the group algebra $k[G]$ as a σ-commutative algebra in the category of G–graded G–modules.

Any braided derivation $f : k[G] \longrightarrow k[G]$ of degree $g \in G$ is determined by some function $\nu : G \longrightarrow k$, where $f(\delta_h) = \nu(h)\,\delta_{gh}$.

From the braided Leibniz rules we get the following relations on the function:

$$\begin{cases} \nu(h_1h_2) = \nu(h_1) + \nu(h_2), \\ \nu(h_1^{-1}h_2h_1) = \nu(h_2), \end{cases}$$

for all $h_1, h_2 \in G$.

Remark that the second condition is a consequence of the first one.

(4) The A–module of derivations of the quantum hyperplane is generated by operators ∂_i of degree -1_i such that $\partial_i(x_j) = \delta_{ij}$.

The Leibniz rule produces the following commutation relations:

$$\partial_i \cdot x_j - \omega_{ij}^{-1} x_j \cdot \partial_i = \delta_{ij}.$$

2.2. Below we collect together the properties of braided derivations.

Theorem.

(1) *$D(P)$ is a σ–symmetric A–module, i.e. $D_\sigma(P) = D(P)$, or*

$$a \cdot f = \sum \gamma'(a) \cdot \gamma''(f),$$

for all $a \in A$, $\quad f \in D(P)$.

(2) *The module $D(A)$ is closed with respect to the braided commutator*

$$[f, g]_\sigma = f \circ g - \sum \sigma'(g) \circ \sigma''(f),$$

and the commutator is an H–invariant:

$$h([f, g]_\sigma) = \sum [h_{(1)}(f), h_{(2)}(g)]_\sigma,$$

for all $f, g \in D(A)$, $\quad h \in H$.

(3) *Let $f_i \in D(A)$, $\quad i = 1, 2, 3$ be braided derivations of the algebra A such that*

$$f_1 \circ f_2 = \sum \gamma'(f_1) \circ \gamma''(f_2), \tag{1}$$

and

$$\sum f_1 \circ \sigma'(f_3) \circ \sigma''(f_2) = \sum \gamma'(f_1) \circ \sigma'(f_3) \circ \sigma''(\gamma''(f_2)). \tag{2}$$

Then the braided Jacobi identity holds:

$$[f_1, [f_2, f_3]_\sigma]_\sigma = [[f_1, f_2]_\sigma, f_3]_\sigma + \sum [\sigma'(f_1), [\sigma''(f_2), f_3]_\sigma]_\sigma. \tag{3}$$

Proof.

(1) Using the definition of braided derivations, we get

$$\sum \gamma'(a)\gamma''(f)(b) = \sum \sigma'\tilde{\sigma}''(a)\, \sigma''\tilde{\sigma}'(f)(b) =$$
$$\sum \tilde{\sigma}'(f)(\tilde{\sigma}''(a)b) - \sum \tilde{\sigma}'(f)(\tilde{\sigma}''(a))\, b =$$
$$a \cdot f(b).$$

(2) The σ– commutator is a braided differential operator of the order 1, and $[f, g]_\sigma(1) = 0$.

Therefore, $[f, g]_\sigma \in D(A)$.

To prove H-invariance of the commutator we have:

$$h([f, g]_\sigma) = h(f \circ g) - \sum h(\sigma'(g) \circ \sigma''(g)) =$$
$$\sum_h h_{(1)}(f) \circ h_{(2)}(g) - \sum_h h_{(1)}\sigma'(g) \circ h_{(2)}\sigma''(f) =$$
$$\sum_h h_{(1)}(f) \circ h_{(2)}(g) - \sum_h \sigma' h_{(2)}(g) \circ \sigma'' h_{(1)}(f) =$$
$$\sum_h [h_{(1)}(f), h_{(2)}(g)]_\sigma.$$

(3) We have

$$[f_1,[f_2,f_3]_\sigma]_\sigma = f_1\circ f_2\circ f_3 - \sum f_1\circ\sigma'(f_3)\circ\sigma''(f_2) -$$
$$\sum \sigma'_{(1)}(f_2)\circ\sigma'_{(2)}(f_3)\circ\sigma''(f_1) + \sum \tilde\sigma'\sigma''_{(2)}(f_3)\circ\tilde\sigma''\sigma'_{(1)}(f_2)\circ\sigma''(f_1),$$
$$[[f_1,f_2]_\sigma,f_3]_\sigma = f_1\circ f_2\circ f_3 - \sum\sigma'(f_2)\circ\sigma''(f_1)\circ f_3 -$$
$$\sum\sigma'(f_3)\circ\sigma''_{(1)}(f_1)\circ\sigma''_{(2)}(f_2) + \sum\sigma'(f_3)\circ\tilde\sigma'\sigma''_{(2)}(f_2)\circ\tilde\sigma''\sigma''_{(1)}(f_1),$$

and

$$\sum[\sigma'(f_2),[\sigma''(f_1),f_3]_\sigma]_\sigma = \sum\sigma'(f_2)\circ\sigma''(f_1)\circ f_3 -$$
$$\sum\sigma'(f_2)\circ\tilde\sigma'(f_3)\circ\tilde\sigma''\sigma''(f_1) - \sum\sigma'_{(1)}\tilde\sigma''(f_1)\circ\sigma'_{(2)}(f_3)\circ\sigma''\tilde\sigma'(f_2) +$$
$$\sum\sigma'\tilde\sigma'_{(2)}(f_3)\circ\sigma''\tilde\sigma'_{(1)}\bar\sigma''(f_1)\circ\tilde\sigma''\bar\sigma'(f_2).$$

Comparing coefficients of terms with $f_i f_j f_k$, $i,j,k=1,2,3$ we see that they are equal in the following cases:

(1) $f_2 f_3 f_1 \Longrightarrow$ by the hexagon equations,
(2) $f_2 f_1 f_3$ and $f_1 f_2 f_3 \Longrightarrow$ are simply equal,
(3) $f_3 f_2 f_1 \Longrightarrow$ by the Yang–Baxter equation.

The rest of the Jacobi identity composed of the terms $f_1 f_3 f_2$ and $f_3 f_1 f_2$ is the following:

$$\sum f_1\circ\sigma'(f_3)\circ\sigma''(f_2) - \sum\sigma'_{(1)}\tilde\sigma''(f_1)\circ\sigma'_{(2)}(f_3)\circ\sigma''\tilde\sigma'(f_2) +$$
$$\sum\sigma'\tilde\sigma'_{(2)}(f_3)\circ\sigma''\tilde\sigma'_{(1)}\bar\sigma''(f_1)\circ\tilde\sigma''\bar\sigma'(f_2) - \sum\sigma'(f_3)\circ\sigma''_{(1)}(f_1)\circ\sigma''_{(2)}(f_2) =$$
$$\sum f_1\circ\sigma'(f_3)\circ\sigma''(f_2) - \sum\gamma'(f_1)\circ\sigma'(f_3)\circ\sigma''\gamma''(f_2) +$$
$$\sum\sigma'(f_3)\circ{\sigma''}_{(1)}\gamma'(f_1)\circ{\sigma''}_{(2)}\gamma''(f_2) - \sum\sigma'(f_3)\circ{\sigma''}_{(1)}(f_1)\circ{\sigma''}_{(2)}(f_2).$$

□

Corollary 1. *The braided Jacobi identity holds if f_1 or f_2 is an H–invariant braided derivation.*

Proof. Suppose, for instance, that f_1 is an H–invariant derivation. Then

$$\sum\gamma'(f_1)\circ\gamma''(f_2) = \sum\varepsilon(\gamma')f_1\circ\gamma''(f_2) = f_1\circ f_2.$$

In the same way we get condition (2). □

Corollary 2. *Let σ be a symmetry in the category. Then*

(1) *The braided Jacobi identity holds for all braided derivations f_1,f_2,f_3.*
(2) $[f_1,f_2]_\sigma = -\sum[\sigma'(f_2),\sigma''(f_1)]_\sigma$.

2.3. In this section we build up the representative object for the functor of braided derivations $D: P \longrightarrow D(P)$. To do this, we look at σ–symmetric bimodule $\Omega^1(A)$, generated by formal elements $a\,db$, where $a,b\in A$, with following relations:

(1) H–action

$$h(a\,db) = \sum_h h_{(1)}(a)\,dh_{(2)}(b),$$

(2) the right A–module structure

$$db \cdot a = \sum \sigma'(a)\, d\sigma''(b)$$

and σ–symmetric relations

$$a\, db = \sum \gamma'(a)\, d\gamma''(b),$$

where $\gamma = \sigma \cdot \tau(\sigma) = \sum \gamma' \otimes \gamma''$, and

(3) the usual differential relations

$$d(a+b) = da + db, \qquad d(ab) = da \cdot b + a \cdot db.$$

Denote by $d : A \longrightarrow \Omega^1(A)$ the operator: $d : a \mapsto da$.

The properties above imply that d is a braided derivation.

Theorem. *For any braided derivation $f : A \longrightarrow P$ there is a σ– homomorphism $\hat{f} : \Omega^1(A) \longrightarrow P$ such that*

$$f = \hat{f} \circ d.$$

The σ–homomorphism $\hat{f}$ is uniquely determined, and the correspondence $f \mapsto \hat{f}$ establishes an isomorphism in the category between $D(P)$ and $Hom_\sigma(A, P_\sigma)$.

Proof. We define $\hat{f}$ as follows

$$\hat{f}(a\, db) = \sum \sigma'(a)\sigma''(f)(b),$$

for all $a, b \in A$.

At first we show that $\hat{}$ is a morphism in the category.

One has

$$h(\hat{f})(a\, db) = \sum_h h_{(1)}(\hat{f}(Sh_{(3)}(a)\, d(Sh_{(2)}b) = h_{(1)} \sum \sigma'(Sh_{(3)}(a))\sigma''(f)(Sh_{(2)}b) =$$
$$\sum h_{(1)}\sigma'(Sh_{(4)}(a)h_{(2)}(\sigma''(f)(Sh_{(3)}b)) = \sum h_{(1)}\sigma'(Sh_{(3)}(a))(h_{(2)}\sigma'')(f)(b) =$$
$$\sum \sigma'(h_{(2)}Sh_{(3)}a)(\sigma'' h_{(1)})(f)(b) = \sum \sigma'(\varepsilon(h_{(2)})a)(\sigma'' h_{(1)})(f)(b) =$$
$$\sum \sigma'(a)(\sigma'' h)(f)(b) = h(\hat{f})(a\, db).$$

Now we check the first property of σ–homomorphisms.

Let $p = c\, db$, then

$$\hat{f}(ap) = \hat{f}(ac\, db) = \sum \sigma'(ac)\sigma''(f)(b) = \sum \sigma'_{(1)}(a)\sigma'_{(2)}(c)\sigma''(f)(b)$$
$$\overset{\text{hexagon}}{=} \sum \tilde{\sigma}'(a)\sigma'(c)\sigma''\tilde{\sigma}''(f)(b) = \sum \tilde{\sigma}'(a)\tilde{\sigma}''(\hat{f})(c\, db) = \sum \sigma'(a)\sigma''(\hat{f})(p).$$

For the second property we have

$$\begin{aligned}
&\sum \sigma'(\hat{f})(\sigma''(a)p) = \sum \sigma'_{(1)}(\hat{f}(S\sigma_{(4)}\sigma''(a) \cdot S\sigma'_{(3)}(c)S\sigma'_{(2)}(b) = \\
&\sum \sigma'_{(1)}(\tilde{\sigma}'(S\sigma'_{(4)}\sigma''(a)S\sigma'_{(3)}(c))\tilde{\sigma}''(f)(S\sigma'_{(2)}(b))) = \\
&\sum \sigma'_{(1)}(f(S\sigma'_{(4)}\sigma''(a) \cdot S\sigma'_{(3)}(c) \cdot S\sigma'_{(2)}(b))) - \\
&\sum \sigma'_{(1)}(f(S\sigma'_{(4)}\sigma''(a)S\sigma'_{(3)}(c)) \cdot S\sigma'_{(2)}(b)) = \\
&\sum \sigma'(f)(\sigma''(a)cb) - \sum \sigma'(f)(\sigma''(a)c) \cdot b = \\
&af(cb) + \sum \sigma'(f)(\sigma''(a))cb - af(c)b - \sum \sigma'(f)(\sigma''(a))cb = \\
&af(cb) - af(c)b = a\sum \sigma'(c)\sigma''(f)(b) = a\hat{f}(c\,db).
\end{aligned}$$

□

2.4. Starting from this point we will build up an algebra of braided differential forms over a σ–commutative algebra A.

The algebra will be a new $\hat{\sigma}$–commutative algebra $\Omega^*(A) = \sum_{i\in\mathbb{N}} \Omega^i(A)$, for some new braiding $\hat{\sigma}$ equipped with some H–invariant braided derivation of degree 1.

The last conditions dictate some restrictions on the braiding $\hat{\sigma}$.

To describe these braidings we look at the category $\mathcal{G}r(\mathcal{C})$ of $\mathbb{N}$ –graded objects over $\mathcal{C}$.

The category has for objects families of objects in $\mathcal{C}$ i.e. $X = \{X_n, n \in \mathbb{N}\}$. and for morphisms $f : X \longrightarrow Y$ families $f = \{f_n, n \in \mathbb{N}\}$ of morphisms $f_n : X_n \longrightarrow Y_n$ in $\mathcal{C}$.

We introduce the usual tensor product in $\mathcal{G}r(\mathcal{C})$:

$$(X \otimes Y)_n = \sum_i X_i \otimes Y_{n-i}.$$

Observe that $\mathcal{G}r(\mathcal{C})$ has modules of internal homomorphisms
$Hom(X,Y)$, where

$$Hom_n(X,Y) = \{f = \{f_n\} | f_n \in Hom(X_i, Y_{i+n}) \forall i, n \in \mathbb{N}\}.$$

We will call elements of $Hom_n(X,Y)$ *internal homomorphisms of degree n.*

Below we identify in the usual way an object Z of the category $\mathcal{C}$ with the object (Z_n) of the category $\mathcal{G}r(\mathcal{C})$, where $Z_0 = Z$, and $Z_n = 0$ otherwise.

In a similar way, we identify morphisms and internal homomorphisms in $\mathcal{C}$ with morphisms and internal homomorphisms in $\mathcal{G}r(\mathcal{C})$.

Theorem. *Any braiding $\hat{\sigma}_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ in the category $\mathcal{G}r(\mathcal{C})$ has the form*

$$\hat{\sigma}_{X,Y}(x_n \otimes y_m) = \hat{\sigma}_{n,m} \cdot (y_m \otimes x_n),$$

for some family $\{\sigma_{n,m}, n, m \in \mathbb{N}\}$ of elements of $H \otimes H$, and where $x_n \in X_n, y_m \in Y_m$.

The family $\{\sigma_{n,m}\}$ is completely determined by the following data:

(1) *a braiding $\sigma = \hat{\sigma}_{0,0}$ in the category $\mathcal{C}$,*
(2) *two invertible central group–like elements $\phi, \psi \in H$,*
(3) *an invertible element $q \in U(k)$,*
(4) *for the data (σ, ϕ, ψ, q) the braiding $\hat{\sigma}$ is given by the formula*

$$\hat{\sigma}_{n,m} = (q)^{nm}(\phi^n \otimes \psi^m) \cdot \sigma. \tag{1}$$

Proof. Rewriting the hexagon conditions for $\hat{\sigma}$ in terms of the family $\hat{\sigma}_{n,m}$ we obtain the following relations :

$$(id \otimes \Delta)\hat{\sigma}_{n+m,k} = (\hat{\sigma}_{n,k} \otimes 1)(\hat{\sigma}_{m,k})_{13}, \tag{2}$$
$$(\Delta \otimes id)\hat{\sigma}_{n,m+k} = (1 \otimes \hat{\sigma}_{n,k})(\hat{\sigma}_{nm})_{13}, \tag{3}$$
$$\hat{\sigma}_{n,m} \cdot \tau(\Delta h) = \Delta(h) \cdot \hat{\sigma}_{n,m}, \quad \forall h \in H; n, m \in \mathbb{N}. \tag{4}$$

Now by applying the morphism $id \otimes \varepsilon \otimes id$ to the both sides of formula (2) we get the following recursive relation:

$$\hat{\sigma}_{n+m,k} = (\phi_{n,k} \otimes 1)\hat{\sigma}_{m,k}, \tag{5}$$

where

$$\phi_{n,k} = (id \otimes \varepsilon)\hat{\sigma}_{n,k}. \tag{6}$$

In the same way we get from (3)

$$\hat{\sigma}_{n,m+k} = (1 \otimes \psi_{n,k})\hat{\sigma}_{n,m}, \tag{7}$$

where

$$\psi_{n,k} = (\varepsilon \otimes id)\hat{\sigma}_{n,k}. \tag{8}$$

Letting $n = 1$ in (5), we get $\hat{\sigma}_{m+1,k} = (\phi_{1,k} \otimes 1)\hat{\sigma}_{m,k}$, and therefore

$$\hat{\sigma}_{m,k} = (\phi_{1,k}^m \otimes 1)\hat{\sigma}_{0,k}.$$

In a similar way, letting $k = 1$ in formula (6), we get $\hat{\sigma}_{n,m+1} = (1 \otimes \psi_{n,1})\hat{\sigma}_{n,m}$, and therefore

$$\hat{\sigma}_{n,m} = (1 \otimes \psi_{n,1}^m)\hat{\sigma}_{n,0}.$$

Let $\phi = \phi_{1,0}, \psi = \psi_{0,1}, \sigma = \sigma_{0,0}$, then $\hat{\sigma}_{n,0} = (\phi^n \otimes 1)\sigma$, and $\hat{\sigma}_{0,k} = (1 \otimes \psi^k)\sigma$.

From the relations

$$\phi_{1,k} = (id \otimes \varepsilon)\hat{\sigma}_{1,k} = \begin{cases} (id \otimes \varepsilon)(\phi_{1,k} \otimes \psi^k)\sigma = \varepsilon(\psi)^k \phi_{1,k}, \\ (id \otimes \varepsilon)(\phi \otimes \psi_{1,1}^k)\sigma = \varepsilon(\psi_{1,1})^k \phi, \end{cases}$$

we get $\varepsilon(\psi) = 1$, and $\phi_{1,k} = \varepsilon(\psi_{1,1})^k \phi$.

In a similar way, from the relations

$$\psi_{n,1} = (\varepsilon \otimes id)\hat{\sigma}_{n,1} = \begin{cases} (\varepsilon \otimes id)(\phi_{1,1}^n \otimes \psi)\sigma = \varepsilon(\phi_{1,1})^n \psi, \\ (\varepsilon \otimes id)(\phi^n \otimes \psi_{n,1})\sigma = \varepsilon(\phi)^n \psi_{n,1}, \end{cases}$$

we get $\varepsilon(\phi) = 1$, and $\psi_{n,1} = \varepsilon(\phi_{1,1})^n \psi$.

Now if we put $q = \varepsilon(\psi_{1,1}) = \varepsilon(\phi_{1,1}) \in k$, we obtain formula (1). By substituting this formula in relations(2),(3) and (4) we find that ϕ and ψ are central group–like elements. $\square$

2.5. To motivate the following considerations, we assume that a σ–commutative algebra A is embedded in some $\hat{\sigma}$–commutative algebra $\hat{A} = \sum_{n \in \mathbb{N}} A_n, \quad A_0 = A$, with wedge multiplication $\wedge$.

Assume also that the algebra $\hat{A}$ is equipped with a non–trivial H– invariant braided derivation d.

The braided Leibniz rule for H–invariant derivations of the algebra takes the form

$$\begin{cases} d(\alpha_n \wedge \alpha_m) = d\alpha_n \wedge \alpha_m + q^n \phi(\alpha_n) \wedge d(\alpha_m), \\ q^n\, d(\psi(\alpha_n) \wedge \alpha_m) = \alpha_n \wedge d(\alpha_m) + q^n\, d(\psi(\alpha_n)) \wedge \alpha_m, \end{cases}$$

where $\alpha_n \in A_n, \alpha_m \in A_m$.

Comparing these relations shows that $q^{2n} \phi\psi = 1$ on $A_n, \quad \forall n \in \mathbb{N}$. Therefore, $q^2 = 1$ and $\phi\psi = 1$ on $\hat{A}$.

We should remark also that $[d, d]_{\hat{\sigma}} = (1 - q)d^2$, and therefore d^2 is a braided derivation if $(1 - q)$ is an invertible element of k.

Definition. The braiding $\hat{\sigma}$ given by formula 2.4.(1) with $q = -1$ and $\phi\psi = 1$ will be called a *differential prolongation* of the braiding σ.

2.6. Let us fix a braiding σ, group–like element $\phi \in H$ and let $\hat{\sigma}$ be the differential prolongation of σ.

Denote by $\Omega^1(A, \phi)$ a bimodule generated by formal elements $a\, d_\phi b$, where $a, b \in A$, with new relations (cf.2.3.)

(1) H–action

$$h(a\, d_\phi b) = \sum_h h_{(1)}(a)\, d_\phi h_{(2)}(b),$$

(2) the right A–module structure

$$d_\phi b \cdot a = \sum \phi\sigma'(a)\, d_\phi \sigma''(b),$$

(3) $\hat{\sigma}$–symmetric relations

$$a\, d_\phi b = \sum \phi\gamma'(a)\, d_\phi \gamma''(b),$$

(4) and new differential relations

$$d_\phi(a + b) = d_\phi(a) + d_\phi(b), \qquad d_\phi(ab) = d_\phi(a) \cdot b + \phi(a)\, d_\phi(b).$$

Note that $\Omega^1(A, 1) = \Omega^1(A)$, when $\phi = 1$ on A.

Denote by

$$\Omega^*(A, \phi) = \sum_{n \in \mathbb{N}} \Omega^n(A, \phi)$$

the $\hat{\sigma}$–commutative algebra generated by A and $\Omega^1(A,\phi)$.

Let $d_\phi : \Omega^n(A,\phi) \longrightarrow \Omega^{n+1}(A,\phi), \quad n \in \mathbb{N}$, be the $\hat{\sigma}$–derivation of degree 1, defined by the formula

$$d_\phi(\alpha \wedge \beta) = d_\phi \alpha \wedge \beta + (-1)^n \phi^n(\alpha) \wedge d_\phi \beta,$$

for all $\alpha \in \Omega^n(A,\phi), \quad \beta \in \Omega^*(A,\phi)$.

Then $d_\phi^2 = 0$, and for any σ–commutative algebra A we get the complex:

$$0 \longrightarrow A \xrightarrow{d_\phi} \Omega^1(A,\phi) \xrightarrow{d_\phi} \cdots \xrightarrow{d_\phi} \Omega^n(A,\phi) \xrightarrow{d_\phi} \Omega^{n+1}(A,\phi) \xrightarrow{d_\phi} \cdots$$

The cohomology of this complex at the term $\Omega^n(A,\phi)$ will be denoted by $H^n(A,\phi)$ and called *braided de Rham cohomology* of the algebra A.

Note that the structure of a $\hat{\sigma}$–multiplicative algebra in $\Omega^*(A,\phi)$ induces the same structure in the braided cohomology algebra

$$H^*(A,\phi) = \sum_{n \in \mathbb{N}} H^n(A,\phi).$$

Examples.

(1) In the category of G–graded modules the construction of the algebra of differential forms over σ–commutative algebra A depends on invertible group-like elements $\phi \in k(G)$.

Therefore the construction is determined by the group homomorphisms $\phi : G \longrightarrow U(k)$.

For instance, for the trivial group $G = \{e\}$ we a unique algebra but for the super-case $G = \mathbb{Z}_2$ we have two algebras of differential forms.

(2) For the case of quantum hyperplane $k = \mathbb{C}, \quad G = \mathbb{Z}^n$, the homomorphisms $\phi : \mathbb{Z}^n \longrightarrow \mathbb{C}^*$ have the form $\phi(a) = z^a$, for some complex vector $z = (z_1, ..., z_n) \in (\mathbb{C}^*)^n$.

The algebra of differential forms for the given ϕ generaded by the elements x_i and dx_j and the relations:

$$x_i x_j = \omega_{ij}\, x_j x_i, \quad x_i\, dx_j = z_i \omega_{ij}\, dx_j\, x_i, \quad dx_i \wedge dx_j = -z_i z_j \omega_{ij}\, dx_j \wedge dx_i.$$

2.7. Let X be a left A–module in the category $\mathcal{C}$ and let $n \in \mathbb{N}$. We denote by $X_{(n)}$ a left $\Omega^*(A,\phi)$–module in the category $\mathcal{G}r(\mathcal{C})$ such that $(X_{(n)})_n = X$, and $(X_{(n)})_k = 0$ otherwise, with obvious multiplication: $\omega \cdot x = 0$, if $\omega \in \Omega^*(A,\phi), \quad deg\, \omega > 1$, and $a \cdot x = ax$, if $x \in X, a \in A$.

As above, we may introduce a right $\Omega^*(A,\phi)$–module structure in $X_{(n)}$:

$$x \cdot a \overset{\text{def}}{=} \sum \phi^n \sigma'(a)\, \sigma''(x),$$

if $a \in A, x \in X$, and $x \cdot \omega = 0$, if $\omega \in \Omega^*(A,\phi), deg\, \omega > 1$.

The following calculation

$$\sum \hat{\sigma}'_{0,n}(x) \hat{\sigma}''_{0,n}(a) = \sum \sigma'(x)\, \phi^{-n} \sigma''(a) \overset{\text{def}}{=}$$
$$\sum \tilde{\sigma}' \sigma''(a)\, \tilde{\sigma}'' \sigma'(x) = \sum \gamma'(a)\, \gamma''(x)$$

shows that $X_{(n)}$ is a $\hat{\sigma}$–symmetric $\Omega^*(A,\phi)$–bimodule if X is a σ–symmetric A–module.

Let $f : \Omega^*(A,\phi) \longrightarrow P$ be a $\hat{\sigma}$–derivation of degree k with values in $\hat{\sigma}$–symmetric $\Omega^*(A,\phi)$–module P.

We consider the restriction $f_0 = f|_A : A \longrightarrow P_k$ as a k–homomorphism of $\hat{\sigma}$–commutative algebra $A = A_{(0)}$ into $\hat{\sigma}$–symmetric $\Omega^*(A,\phi)$–module $(P_k)_{(k)}$.

These restrictions may be characterized by a new Leibniz rule.

Definition. Let X be a σ–symmetric left A–module. An internal homomorphism $f : A \longrightarrow X$ will be called *twisted* (or *ϕ– twisted*) *derivation* of degree $k \in \mathbb{Z}$, if the following *twisted Leibniz rule* holds:

$$\begin{cases} f(ab) = f(a)\, b + \sum \phi^k \sigma'(a)\, \sigma''(f)(b), \\ \sum \sigma'(f)(\phi^{-k}\sigma''(a) b) = a\, f(b) + \sum \sigma'(f)(\phi^{-k}\sigma''(a))\, b \end{cases}$$

Denote by $D_{\phi,k}$ the module of all twisted derivations of degree k.

Remarks.

(1) We have $f_0 \in D_{\phi,k}(P_k)$, for the restriction f_0.

(2) If $f : A \longrightarrow P$ is an H–invariant twisted derivation of degree k, then the twisted Leibniz rule takes the form:

$$f(ab) = f(a)\, b + \phi^k(a) f(b).$$

(3) The differentials $d_{\phi^k} : A \longrightarrow \Omega^1(A,\phi^k)$ are twisted H–invariant derivations of degree k.

Theorem. *The morphisms $d_{\phi^k} : A \longrightarrow \Omega^1(A,\phi^k)$ are universal twisted derivations of degree k in the following sense.*

Any twisted derivation $f : A \longrightarrow X$ of degree k may be represented as the composition $f = \bar{f} \circ d_{\phi^k}$, where $\bar{f} : \Omega^1(A,\phi^k) \longrightarrow X$ determines $\hat{\sigma}$–homomorphism $(\Omega^1(A,\phi^k))_{(k)} \longrightarrow X_{(k)}$. The correspondence $f \mapsto \bar{f}$ establishes an isomorphism

$$D_{\phi,k}(X) \simeq Hom_{\hat{\sigma},0}(\Omega^1(A,\phi^k)_{(k)}, X_{(k)}) \simeq Hom_{\sigma}(\Omega^1(A,\phi^k), X).$$

2.8. We say that a $\hat{\sigma}$–derivation $\Omega^*(A,\phi) \longrightarrow P$ is *algebraic*, if $f|_A = 0$.

Denote by $D_k^{\mathrm{alg}}(P)$ the $\Omega^*(A,\phi)$–module of all the algebraic derivations of degree $k \in \mathbb{Z}$, and by $D_*^{\mathrm{alg}}(P) = \sum_{k\in\mathbb{Z}} D_k^{\mathrm{alg}}(P)$ the graded module of all the algebraic derivations.

Remark that any algebraic derivation is determined by its restriction to $\Omega^1(A,\phi)$. Therefore we have $D_k^{\mathrm{alg}}(P) = 0$, if $k < -1$.

Let $\Omega^1_{\mathrm{alg},k}(\Omega^*(A,\phi))$ be a representative object for the functor of algebraic derivations of degree k, and let $\partial_k^a : \Omega^*(A,\phi) \longrightarrow \Omega^1_{\mathrm{alg},k}(\Omega^*(A,\phi))$ be the universal algebraic derivation.

As before, we may consider $\Omega^1_{\mathrm{alg},k}(\Omega^*(A,\phi))$ as a $\hat{\sigma}$–symmetric module generated by formal elements $\alpha \wedge \partial_k^a \beta$, where $\alpha, \beta \in \Omega^*(A,\phi)$, with relations

(1) H–action

$$h(\alpha \wedge \partial_k^a \beta) = \sum_h h_{(1)}(\alpha) \wedge \partial_k^a h_{(2)}(\beta),$$

(2) the right $\Omega^*(A,\phi)$–module structure

$$\partial_k^a \beta \wedge \alpha = \sum \phi^k \sigma'(\alpha) \wedge \partial_k^a \sigma''(\beta),$$

and the $\hat{\sigma}$–symmetric relations

$$\alpha \wedge \partial_k^a \beta = \sum \gamma'(\alpha) \wedge \partial_k^a \gamma''(\beta),$$

(3) algebraic differential relations

$$\partial_k^a(\alpha+\beta) = \partial_k^a \alpha + \partial_k^a \beta, \qquad \partial_k^a(\alpha\wedge\beta) = \partial_k^a(\alpha)\wedge\beta + \phi^k(\alpha)\wedge\partial_k^a(\beta),$$

and

$$\partial_k^a(x) = 0,$$

for all $x \in A$.

Let P be a $\Omega^*(A,\phi)$–module in the category $\mathcal{G}r(\mathcal{C})$. We will denote by $P_{(k)}$ the shifted module: $(P_{(k)})_n = P_{n+k}, \quad n \in \mathbb{Z}$.

Theorem. *Any algebraic derivation $f : \Omega^*(A,\phi) \longrightarrow P$, of degree $k \geq -1$, may be represented in a unique way as a composition $f = \bar{f} \circ \partial_k^a$, where*

$$\bar{f} : \Omega^1_{\mathrm{alg},k}(\Omega^*(A,\phi)) \longrightarrow P$$

determines $\hat{\sigma}$–homomorphism $(\Omega^1_{\mathrm{alg},k}(\Omega^*(A,\phi)))_{(k)} \longrightarrow P_{(k)}$ and the map $f \mapsto \bar{f}$ establishes an isomorphism

$$D_k^{\mathrm{alg}}(P) \simeq Hom_{\hat{\sigma},0}(\Omega^1_{\mathrm{alg},k}, P) \simeq Hom_{\hat{\sigma},0}(\Omega^1_{\mathrm{alg},k}, P_{(k)}).$$

2.9. Now we look at the algebra $\Omega^*(A,\phi)$ as a new $\hat{\sigma}$–commutative algebra and build up the new universal module of braided differential forms $\Omega^1(\Omega^*(A,\phi))$ in the category $\mathcal{G}r(\mathcal{C})$ together with new universal $\hat{\sigma}$–derivation

$$\partial : \Omega^*(A,\phi) \longrightarrow \Omega^1(\Omega^*(A,\phi)),$$

of degree 0.

We may consider $\Omega^1(\Omega^*(A,\phi))$ as a $\hat{\sigma}$–symmetric $\mathbb{Z}$–graded module

$$\Omega^1(\Omega^*(A,\phi)) = \sum_k \Omega^1(\Omega^*(A,\phi))_k,$$

where $\Omega^1(\Omega^*(A,\phi))_k$ is generated by formal elements $\alpha\wedge\partial_k\beta$, where $\alpha,\beta \in \Omega^*(A,\phi)$, with following relations:

(1) H–action

$$h(\alpha\wedge\partial_k\beta) = \sum_h h_{(1)}(\alpha)\wedge\partial_k h_{(2)}(\beta),$$

(2) the right $\Omega^*(A,\phi)$–module structure

$$\partial_k\beta\wedge\alpha = \sum \phi^k\sigma'(\alpha)\wedge\partial_k\sigma''(\beta),$$

and the $\hat{\sigma}$–symmetric relations

$$\alpha\wedge\partial_k\beta = \sum \gamma'(\alpha)\wedge\partial_k\gamma''(\beta),$$

(3) twisted differential relations

$$\partial_k(\alpha+\beta) = \partial_k\alpha + \partial_k\beta, \quad \partial_k(\alpha\wedge\beta) = \partial_k\alpha\wedge\beta + \phi^k(\alpha)\wedge\partial_k\beta.$$

Summarizing, we obtain the following

Theorem. *The pair* $\left(\Omega^1(\Omega^*(A,\phi)), \partial = \sum \partial_k\right)$ *is a representative object for the functor of graded braided derivations of the algebra* $\Omega^*(A,\phi)$, *and*

(1) *the restriction map* $f \in D_k(P) \mapsto f|_A \in D_{\phi,k}(P_k)$ *defines an embedding*

$$0 \longrightarrow \Omega^1(A,\phi^k) \longrightarrow \Omega^1(\Omega^*(A,\phi))_k,$$

where $a\, d_{\phi^k} b \mapsto a\, \partial_k b, \quad a, b \in A$,

(2) *the embedding* $D_*^{\mathrm{alg}}(P) \subset D_*(P)$ *defines epimorphisms*

$$\Omega^1(\Omega^*(A,\phi))_k \longrightarrow \Omega^1_{\mathrm{alg},k}(\Omega^*(A,\phi)) \longrightarrow 0,$$

such that $\alpha \wedge \partial_k \beta \mapsto \alpha \wedge \partial_k^a \beta$,

(3) *the sequence*

$$0 \longrightarrow \Omega^1(A,\phi^k) \longrightarrow \Omega^1(\Omega^*(A,\phi))_k \longrightarrow \Omega^1_{\mathrm{alg},k}(\Omega^*(A,\phi)) \longrightarrow 0$$

is exact.

2.10. In this section we describe the module of braided $\hat{\sigma}$– derivations of the algebra braided differential forms $\Omega^*(A,\phi)$.

We start with an explicit description of algebraic braided derivations. Any such derivation $f : \Omega^*(A,\phi) \longrightarrow \Omega^*(A,\phi)$ of degree k is completely determined by the restriction on $\Omega^1(A,\phi)$ and therefore we obtain an isomorphism:

$$D_k^{\mathrm{alg}}(\Omega^*(A,\phi)) \simeq Hom_\sigma(\Omega^1(A,\phi), \Omega^{k+1}(A,\phi)).$$

The image $\imath_\lambda \in D_k^{\mathrm{alg}}(\Omega^*(A,\phi))$ of the element $\lambda \in Hom_\sigma(\Omega^1(A,\phi), \Omega^{k+1}(A,\phi))$ will be called *inner braided derivation.* One can define $\imath_\lambda$ directly using the braided Leibniz rule:

the derivations

$$\imath_\lambda : \Omega^j(A,\phi) \longrightarrow \Omega^{j+k}(A,\phi)$$

are determined by the following relations:

(1) $\imath_\lambda(a) = 0, \quad \forall a \in A = \Omega^0(A,\phi)$,
(2) $\imath_\lambda(\omega) = \lambda(\omega), \quad \forall \omega \in \Omega^1(A,\phi)$,
(3) $h(\imath_\lambda) = \imath_{h(\lambda)}, \quad \forall h \in H$,
(4) $\imath_\lambda(\omega_1 \wedge \omega_2) = \imath_\lambda(\omega_1) \wedge \omega_2 + (-1)^{kj} \sum \phi^k \sigma'(\omega_1) \wedge \phi^{-j} \sigma''(\omega_2)$,
(5) $\sum \phi^j \imath_{\sigma'(\lambda)}(\phi^{-k}\sigma''(\omega_1) \wedge \omega_2) = \omega_1 \wedge \imath_\lambda(\omega_2) + \sum \phi^j \imath_{\sigma'(\lambda)}(\phi^{-k}\sigma''(\omega_1)) \wedge \omega_2$,

for all $\omega_1 \in \Omega^j(A,\phi), \omega_2 \in \Omega^*(A,\phi)$.

Denote by

$$\mathcal{N}_k(A,\phi) = Hom_\sigma(\Omega^1(A,\phi), \Omega^{k+1}(A,\phi))$$

and let

$$\mathcal{N}_*(A,\phi) = \sum_{k \in \mathbb{Z}} \mathcal{N}_k(A,\phi).$$

We will call this module a *Nijenhuis algebra* of the σ–commutative algebra A.

The discussion above shows that we have isomorphisms

$$\mathcal{N}_k(A,\phi) \simeq D_k{}^{\mathrm{alg}}(\Omega^*(A,\phi)).$$

Modules of braided derivations are closed with respect to braided commutators.

Therefore we obtain a bilinear structure:

$$[,]_{\hat\sigma} : \mathcal{N}_k(A,\phi) \times \mathcal{N}_l(A,\phi) \longrightarrow \mathcal{N}_{k+l}(A,\phi),$$

in the $\mathcal{N}_*(A,\phi)$.

Here we define $[\lambda_1,\lambda_2]_{\hat\sigma}$ from the following relation:

$$\imath_{[\lambda_1,\lambda_2]_{\hat\sigma}} = [\imath_{\lambda_1},\imath_{\lambda_2}]_{\hat\sigma}.$$

The bracket $[,]_{\hat\sigma}$ will be called *braided algebraic Nijenhuis bracket.*

From the definition of the bracket we have:

$$[\lambda_1,\lambda_2]_{\hat\sigma} = \imath_{\lambda_1}(\lambda_2(\omega)) - (-1)^{kl}\sum \imath_{\phi^k\sigma'(\lambda_2)}(\phi^{-l}\sigma''(\lambda_1)),$$

where $\omega \in \Omega^1(A,\phi)$ $\quad \lambda_1 \in \mathcal{N}_k(A,\phi), \lambda_2 \in \mathcal{N}_l(A,\phi)$.

2.11. A *braided Lie derivation* $\mathcal{L}_\lambda$ with respect to element $\lambda \in \mathcal{N}_k(A,\phi)$ will mean the braided derivation

$$\mathcal{L}_\lambda = [d_\phi, \imath_\lambda]_{\hat\sigma}.$$

Note that $\mathcal{L}_\lambda \in D_{k+1}(\Omega^*(A,\phi))$, if $\lambda \in \mathcal{N}_k(A,\phi)$.

Below we collect together the main properties of braided Lie derivations.

Theorem. *The braided Lie derivations are $\hat\sigma$–derivations of the $\hat\sigma$–commutative algebra $\Omega^*(A,\phi)$ of braided differential forms such that:*

1. $[d_\phi, \mathcal{L}_\lambda]_{\hat\sigma} = 0, \quad \forall\lambda \in \mathcal{N}_*(A,\phi)$,
2. $h(\mathcal{L}_\lambda) = \mathcal{L}_{h(\lambda)}, \quad \forall h \in H$,
3. *The braided commutator $[\mathcal{L}_{\lambda_1},\mathcal{L}_{\lambda_2}]_{\hat\sigma}$ is a braided Lie derivation $\mathcal{L}_{\{\lambda_1,\lambda_2\}}$ for some element $\{\lambda_1,\lambda_2\}$. This element is called a braided differential Frolicher–Nijenhuis bracket.*
4. *Any $\hat\sigma$–derivation $f : \Omega^*(A,\phi) \longrightarrow \Omega^*(A,\phi)$ of the algebra braided differential forms $\Omega^*(A,\phi)$ may be represented as follows:*

   $$f = \imath_{\lambda_1} + \mathcal{L}_{\lambda_2},$$

   for some uniquely determined elements λ_1 and λ_2.

   Hence, there is a decomposition:

   $$D_*(\Omega^*(A,\phi)) = D_*{}^{\mathrm{alg}}(\Omega^*(A,\phi)) \oplus D_*{}^{\mathrm{Lie}}(\Omega^*(A,\phi)),$$

   such that

   $$\left[D_*{}^{\mathrm{Lie}}(\Omega^*(A,\phi)), D_*{}^{\mathrm{Lie}}(\Omega^*(A,\phi))\right]_{\hat\sigma} \subset D_*{}^{\mathrm{Lie}}(\Omega^*(A,\phi)),$$

   $$\left[D_*{}^{\mathrm{alg}}(\Omega^*(A,\phi)), D_*{}^{\mathrm{alg}}(\Omega^*(A,\phi))\right]_{\hat\sigma} \subset D_*{}^{\mathrm{alg}}(\Omega^*(A,\phi)),$$

and

$$\left[d_\phi, D_*{}^{\mathrm{alg}}(\Omega^*(A,\phi))\right]_{\hat\sigma} \subset D_*{}^{\mathrm{Lie}}(\Omega^*(A,\phi)), \qquad \left[d_\phi, D_*{}^{\mathrm{Lie}}(\Omega^*(A,\phi))\right]_{\hat\sigma} = 0.$$

(5) *The following braided analog of the infinitesimal Stokes theorem holds:*

$$\begin{cases} [\imath_{\lambda_1}, \mathcal{L}_{\lambda_2}]_{\hat\sigma} = \imath_{\{\lambda_1,\lambda_2\}} + (-1)^l \mathcal{L}_{\lambda_1 \bullet \lambda_2}, \\ [\mathcal{L}_{\lambda_2}, \imath_{\lambda_1}]_{\hat\sigma} = (-1)^{l+1} \imath_{\{\lambda_2,\lambda_1\}} + (-1)^{kl} \mathcal{L}_{\sum \phi^{k+1}\sigma'(\lambda_1)\bullet\phi^{-l}\sigma''(\lambda_2)} \end{cases}$$

where $\lambda_1 \in \mathcal{N}_k(A,\phi), \lambda_2 \in \mathcal{N}_l(A,\phi)$.

Proof. Properties (1)–(3) of the braided Lie derivations are consequences of properties 2.2. of braided derivations. We should remark only that d_ϕ is an H–invariant braided derivation.

To prove (4), we should note that λ_2 is determined by the restriction $f|_A$. Then $f - \mathcal{L}_{\lambda_2}$ is an algebraic braided derivation, and therefore $f - \mathcal{L}_{\lambda_1} = \imath_{\lambda_2}$.

Note also, that $[d_\phi, f]_{\hat\sigma} = \mathcal{L}_{\lambda_2}$.

To prove (5), we decompose $[\imath_{\lambda_1}, \mathcal{L}_{\lambda_2}]_{\hat\sigma}$ as above:

$$[\imath_{\lambda_1}, \mathcal{L}_{\lambda_2}]_{\hat\sigma} = \imath_x + \mathcal{L}_y,$$

for some elements $x, y \in \mathcal{N}_*(A,\phi)$. To determine y, we should look at the restriction $[\imath_{\lambda_1}, \mathcal{L}_{\lambda_2}]_{\hat\sigma}$ on A.

One gets

$$[\imath_{\lambda_1}.\mathcal{L}_{\lambda_2}]_{\hat\sigma}(a) = \imath_{\lambda_1} \circ \mathcal{L}_{\lambda_2}(a) = (-1)^{l+1} \imath_{\lambda_1} \circ \imath_{\lambda_2}(d_\phi a) = (-1)^{l+1}(\lambda_1 \bullet \lambda_2)(d_\phi a).$$

Therefore, $y = (-1)^{l+1}\lambda_1 \bullet \lambda_2$.

Moreover, one has

$$\mathcal{L}_x = [d_\phi, \imath_x]_{\hat\sigma} = [d_\phi, [\imath_{\lambda_1}, \mathcal{L}_{\lambda_2}]_{\hat\sigma}]_{\hat\sigma} = [\mathcal{L}_{\lambda_1}, \mathcal{L}_{\lambda_2}]_{\hat\sigma} = \mathcal{L}_{\{\lambda_1,\lambda_2\}}.$$

In a similar one proves the second relation. □

2.12. In this secxtion we define a Lie structure on the modules of braided derivations. There are several definitions of braided Lie algebras ([Gu],[Mj]).

All of them are based on the translation of the Jacobi identity into the framework of braided categories.

Here we are suggesting to change our paradigm and to consider a braided Lie coalgebra structure in modules of braided differential forms instead of Lie algebra structure in modules of braided derivations. This makes it possible to preserve some analogue of the skew symmetry property and write down Jacobi identity as a braided version of the Master equation.

We should point out that our approach is based on the definition of Lie coalgebra structures as invariant braided derivations and we therefore may exploit the theory of braided derivations developed in this section.

Let A be a σ–commutative algebra in the category $\mathcal{C}$ and let M be an $A - A$ bimodule. We fix some differential prolongation $\hat\sigma$ determined by some central group–like element $\phi \in H$.

We will say that M is a $\hat{\sigma}$*–symmetric bimodule* if $M_{(1)}$ is a $\hat{\sigma}$–symmetric $\Omega^*(A,\phi)$–bimodule.

In other words M is a $\hat{\sigma}$–symmetric bimodule if

$$a\,m = \sum \sigma'(m) \cdot \phi^{-1}\sigma''(a),$$

and

$$m\,a = \sum \phi\sigma'(a) \cdot \sigma''(m),$$

for all $a \in A, \quad m \in M$.

Denote by $\Lambda^*_{\hat{\sigma}}(M) = \sum_{n\in\mathbb{N}} \Lambda^n_{\hat{\sigma}}(M)$ the $\hat{\sigma}$–commutative $\mathbb{N}$– graded algebra generated by $\Lambda^0_{\hat{\sigma}}(M) = A$ and $\Lambda^1_{\hat{\sigma}}(M) = M$. Let $\wedge$ be a product in the algebra.

As above we denote by $D^{\mathrm{alg}}_*(\Lambda^*_{\hat{\sigma}}(M)) = \sum_{k\geq -1} D^{\mathrm{alg}}_k(\Lambda^*_{\hat{\sigma}}(M))$ the module of algebraic braided derivations of the algebra. Here we denote by $D^{\mathrm{alg}}_k(\Lambda^*_{\hat{\sigma}}(M))$ module of $\hat{\sigma}$–derivations $f : \Lambda^*_{\hat{\sigma}}(M) \longrightarrow \Lambda^*_{\hat{\sigma}}(M)$ of degree $k \geq -1$, such that $f|_A = 0$.

Let

$$\mathcal{N}_k(M) = Hom_{\hat{\sigma}}(M, \Lambda^{k+1}_{\hat{\sigma}}(M))$$

be a *Nijenhuis module* and

$$\mathcal{N}_*(M) = \sum_{k\geq -1} \mathcal{N}_k(M)$$

be a *Nijenhuis algebra* of the bimodule M.

As we saw above, there is an isomorphism between $\mathcal{N}_k(M)$ and $D^{\mathrm{alg}}_k(\Lambda^*_{\hat{\sigma}}(M))$, given by inner braided derivations: $\alpha \in \mathcal{N}_k(M) \mapsto \imath_\alpha \in D^{\mathrm{alg}}_k(\Lambda^*_{\hat{\sigma}}(M))$.

We have two algebraic structures in $\mathcal{N}_*(M)$:

(1) an associative graded algebra structure

$$\bullet : \mathcal{N}_k(M) \otimes \mathcal{N}_l(M) \longrightarrow \mathcal{N}_{k+l}(M),$$

$$\alpha \otimes \beta \mapsto \alpha \bullet \beta,$$

where

$$(\alpha \bullet \beta)(m) = \imath_\alpha(\beta(m)),$$

for all $m \in M$, and

(2) an algebraic Nijenhuis bracket

$$[.,.] : \mathcal{N}_k(M) \otimes \mathcal{N}_l(M) \longrightarrow \mathcal{N}_{k+l}(M),$$

$$\alpha \otimes \beta \mapsto [\alpha,\beta]_{\hat{\sigma}},$$

where

$$\imath_{[\alpha,\beta]_{\hat{\sigma}}} = [\imath_\alpha, \imath_\beta]_{\hat{\sigma}},$$

or

$$[\alpha,\beta]_{\hat{\sigma}} = (\alpha \bullet \beta) - (-1)^{kl} \sum \phi^k\sigma'(\beta) \bullet \phi^{-l}\sigma''(\alpha).$$

Example. Let $k=-1$. Then $\alpha \in \mathcal{N}_{-1}(M)=M^*$ determines the braided inner derivation or a *braided annihilator operator* $\imath_\alpha : \Lambda^n_{\hat\sigma}(M) \longrightarrow \Lambda^{n-1}_{\hat\sigma}(M)$. Because $\mathcal{N}_{-2}(M)=0$ we get the following commutative relations:

$$[\imath_\alpha, \imath_\beta]_{\hat\sigma} = 0,$$

for all $\alpha, \beta \in \mathcal{N}_{-1}(M)$.

2.13.Definition. We say that a $\hat\sigma$–derivation ∇ of the $\hat\sigma$-commutative algebra $\Lambda^*_{\hat\sigma}(M)$ of degree 1 is a *braided Lie coalgebra structure* on the bimodule M if

(1) ∇ is an H–invariant $\hat\sigma$–derivation,
(2) the *Master equation*
$$[\nabla, \nabla]_{\hat\sigma} = 0, \tag{1}$$

holds.

Therefore a Lie coalgebra structure determines two morphisms

(1) a twisted derivation $\nabla : A \longrightarrow M$, and
(2) a braided symmetric co-product $\nabla : M \longrightarrow \Lambda^2_{\hat\sigma}(M)$,

such that the Master equation holds.

Our main example of a Lie coalgebra structure is the following

Theorem. *The algebra braided differential forms $\Omega^*(A, \phi)$ over σ– commutative algebra A is a braided Lie coalgebra with respect to the braided differential d_ϕ.*

Proof. $[d_\phi, d_\phi] = 2d_\phi^2 = 0$. □

2.14. Let ∇ be a braided Lie coalgebra structure on a bimodule M. Then we can define a ∇–*Lie derivation* $\mathcal{L}^\nabla_\alpha : \Lambda^*_{\hat\sigma}(M) \longrightarrow \Lambda^*_{\hat\sigma}(M)$ for any $\alpha \in \mathcal{N}_k(M)$, as above: $\mathcal{L}^\nabla_\alpha = [\nabla, \imath_\alpha]_{\hat\sigma} \in D_{k+1}(\Lambda^*_{\hat\sigma}(M))$.

Theorem. *The braided ∇-Lie derivations are $\hat\sigma$–derivations of the $\hat\sigma$–commutative algebra $\Lambda^*_{\hat\sigma}(M)$ such that:*

(1) $[\nabla, \mathcal{L}^\nabla_\alpha]_{\hat\sigma} = 0, \quad \forall \alpha \in \mathcal{N}_*(M)$.
(2) $h(\mathcal{L}^\nabla_\alpha) = \mathcal{L}_{h(\alpha)^\nabla}, \quad \forall h \in H$.
(3) *The braided commutator $[\mathcal{L}^\nabla_\alpha, \mathcal{L}^\nabla_\beta]_{\hat\sigma}$ is a braided ∇–Lie derivation $\mathcal{L}_{\{\alpha,\beta\}^\nabla}$ for some element $\{\alpha, \beta\}^\nabla$. The element is called a braided ∇–differential Frolicher–Nijenhuis bracket.*

Proof. See the proof of Theorem 2.10. □

3.Quantizations

In this chapter we define quantizations of monoidal categories and functors between them. We suggest two descriptions of quantizations: one in terms of *quantizers* (and nonlinear cohomologies linked with the description), and the other in terms of *Hochschild cohomologies of Grothendieck ring* of the given monoidal category.

We show that quantizations "deform" all algebraic and differential objects in the category and produce in this way quantizations of the braided differential operators.

3.1. Let $\mathcal{C}$ be a monoidal category equipped with

(1) a bifunctor of tensor product

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

(2) an associativity constraint

$$\alpha = \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z,$$

(3) a unit object $k = k_{\mathcal{C}}$ with natural isomorphisms

$$\eta^l : k \otimes X \longrightarrow X, \qquad \eta^r : X \otimes k \longrightarrow X,$$

such that the following MacLane coherence conditions hold [McL]:

(i) the pentagon axiom:

$$\begin{array}{ccccc} X \otimes (Y \otimes (Z \otimes T)) & \xrightarrow{\alpha} & (X \otimes Y) \otimes (Z \otimes T) & \xrightarrow{\alpha} & ((X \otimes Y) \otimes Z) \otimes T \\ {\scriptstyle id\otimes\alpha}\uparrow & & & & \uparrow{\scriptstyle \alpha\otimes id} \\ X \otimes ((Y \otimes Z) \otimes T) & = & X \otimes ((Y \otimes Z) \otimes T) & \xrightarrow{\alpha} & (X \otimes (Y \otimes Z)) \otimes T \end{array}$$

(ii) the unity axiom:

$$(id \otimes \eta^l) \circ \alpha_{X,k,Y} = \eta^r \otimes id.$$

3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories. We will denote the tensor product in $\mathcal{A}$ by $\otimes$ and that in $\mathcal{B}$ by $\hat{\otimes}$ and associativity and units constraints by $\alpha_{\mathcal{A}}, \eta_{\mathcal{A}}$, and by $\alpha_{\mathcal{B}}, \eta_{\mathcal{B}}$ respectively.

Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unit preserving functor.

Definition.

(1) We say that a natural isomorphism $\mathcal{Q}$

$$\mathcal{Q}_{X,Y} : \Phi(X)\hat{\otimes}\Phi(Y) \longrightarrow \Phi(X \otimes Y), \qquad X, Y \in \mathcal{O}b(\mathcal{A}),$$

is a *quantization* of the functor Φ if $\mathcal{Q}$ preserves a unit object:

$$\Phi(\eta^r_{\mathcal{A}}) \circ \mathcal{Q}_{X,k} = \eta^r_{\mathcal{B}}, \qquad \Phi(\eta^l_{\mathcal{A}}) \circ \mathcal{Q}_{k,X} = \eta^l_{\mathcal{B}},$$

and the following diagram

$$\begin{array}{ccccc} \Phi(X)\hat{\otimes}(\Phi(Y)\hat{\otimes}\Phi(Z)) & \xrightarrow{id\hat{\otimes}\mathcal{Q}} & \Phi(X)\hat{\otimes}\Phi(Y \otimes Z) & \xrightarrow{\mathcal{Q}} & \Phi(X \otimes (Y \otimes Z)) \\ {\scriptstyle \alpha_{\mathcal{B}}}\downarrow & & & & \downarrow{\scriptstyle \Phi(\alpha_{\mathcal{A}})} \\ (\Phi(X)\hat{\otimes}\Phi(Y))\hat{\otimes}\Phi(Z) & \xrightarrow{\mathcal{Q}\hat{\otimes}id} & \Phi(X \otimes Y)\hat{\otimes}\Phi(Z) & \xrightarrow{\mathcal{Q}} & \Phi((X \otimes Y) \otimes Z) \end{array}$$

commutes.

(2) We say that $\mathcal{Q}$ is a *quantization of monoidal category* $\mathcal{A}$ if $\mathcal{Q}$ is a quantization of the identity functor $\Phi = id : \mathcal{A} \longrightarrow \mathcal{A}$.

We show that the above commutative diagram is a realization of some type of MacLane coherence conditions.

To do this, we introduce two monoidal categories. The first one is the category [see McL] of binary words. The words are generated by two symbols: e_0–empty word and (-)–placeholder. We convert the set of binary words into a category $\mathcal{W}$ by introducing one arrow between any binary words of the same length. It is a monoidal category under multiplication of binary words with unit object e_0.

The second category is the monoidal category $\mathcal{I}t(\mathcal{A},\mathcal{B})$ of iterates. Objects of the category are pairs (n,T), where $T:\mathcal{A}^n \longrightarrow \mathcal{B}$ is a functor. Arrows in the category $f:(n.T)\longrightarrow(n,T')$ are natural transformations $f:T\stackrel{\bullet}{\longrightarrow}T'$, [cf.McL]. We convert $\mathcal{I}t(\mathcal{A},\mathcal{B})$ into a monoidal category by introducing multiplication $(m,S)\hat{\otimes}(n,T)=(n+m,S\hat{\otimes}T)$, where $S\hat{\otimes}T$ is the composite

$$S\hat{\otimes}T:\mathcal{A}^{n+m}=\mathcal{A}^n\times\mathcal{A}^m\xrightarrow{S\times T}\mathcal{B}\times\mathcal{B}\xrightarrow{\hat{\otimes}}\mathcal{B}.$$

Any binary word w of length n determines a functor $\Phi_*(w):\mathcal{A}^n\longrightarrow\mathcal{B}$ obtained by replacing each placeholder (-) by the functor Φ.

More precisely, the definition is given by recursion: if we determined functors $\Phi_*(w_1)$ and $\Phi_*(w_2)$ for binary words w_1,w_2 of respective lengths n and m, then $\Phi_*(w_1\cdot w_2)$ is determined by the following diagram

$$\begin{tikzcd}
\mathcal{A}^{n+m}=\mathcal{A}^n\times\mathcal{A}^m \arrow[r, "\Phi_*(w_1\cdot w_2)"] \arrow[d, "\Phi(w_1)\times\Phi(w_2)"'] & \mathcal{B} \\
\mathcal{B}\times\mathcal{B} \arrow[r, "\mathcal{Q}"] & \mathcal{B}\times\mathcal{B}. \arrow[u, "\hat{\otimes}"']
\end{tikzcd}$$

Definition. [cf.Ep] We say that monoidal categories $\mathcal{A}$ and $\mathcal{B}$ are Φ–*coherent* if $\Phi_*:\mathcal{W}\longrightarrow\mathcal{I}t(\mathcal{A},\mathcal{B})$ is a tensor product preserving functor.

The proof of the following theorem is analogous to the proof of theorem 1.6.[Ep].

Theorem. *Monoidal categories $\mathcal{A}$ and $\mathcal{B}$ are Φ–coherent if and only if $\mathcal{Q}$ is a quantization of the functor Φ.*

There is an action of natural isomorphisms of the category $\mathcal{A}$ on the set of all quantizations.

Let $\lambda:\mathcal{A}\stackrel{\bullet}{\longrightarrow}\mathcal{A}$ be a unit preserving natural isomorphism, $\lambda_X:X\longrightarrow X$, and let $\mathcal{Q}$ be a quantization of the functor $\Phi:\mathcal{A}\longrightarrow\mathcal{B}$. Then we can build up a new isomorphism $\mathcal{Q}_\lambda:\Phi(X)\hat{\otimes}\Phi(Y)\longrightarrow\Phi(X\otimes Y)$ by using the following commutative diagram

$$\begin{tikzcd}
\Phi(X)\hat{\otimes}\Phi(Y) \arrow[r, "\mathcal{Q}_{X,Y}"] \arrow[d, "\Phi(\lambda_X)\hat{\otimes}\Phi(\lambda_Y)"'] & \Phi(X\otimes Y) \arrow[d, "\Phi(\lambda_{X\otimes Y})"'] \\
\Phi(X)\hat{\otimes}\Phi(Y) \arrow[r, "(\mathcal{Q}_\lambda)_{X,Y}"] & \Phi(X\otimes Y).
\end{tikzcd}$$

Denote by $\mathcal{Q}(\Phi)$ the set of all the quantizations of the functor Φ and by $\mathcal{Q}H^2(\Phi)$ the set of equivalence classes with respect to the following equivalence relation:

$$\mathcal{Q}\sim\mathcal{Q}'\Leftrightarrow\exists\,\text{a unit preserving transformation}\,\lambda:\mathcal{A}\stackrel{\bullet}{\longrightarrow}\mathcal{A},\text{such that}\,\mathcal{Q}'=\mathcal{Q}_\lambda.$$

The set $\mathcal{Q}H^2(\Phi)$ is an analogue of the non–linear second cohomology group associated to the functor Φ (see 3.6. below).

3.3. Quantizations transport all natural algebraic structures related to tensor product.

Below we list the transport of main structures.

1. Braidings and symmetries.

Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a faithful functor and let σ be a braiding or a symmetry in a monoidal category $\mathcal{A}$.

Then the natural isomorphism $\sigma_{\mathcal{Q}}$, given by the commutative diagram

$$\begin{array}{ccc} \Phi(X)\hat{\otimes}\Phi(Y) & \xrightarrow{\mathcal{Q}_{X,Y}} & \Phi(X \otimes Y) \\ {\scriptstyle \sigma_{\mathcal{Q}}}\big\downarrow & & {\scriptstyle \Phi(\sigma)}\big\downarrow \\ \Phi(Y)\hat{\otimes}\Phi(X) & \xrightarrow{\mathcal{Q}_{Y,X}} & \Phi(Y \otimes X) \end{array}$$

is a braiding or, respectively, symmetry on the image of the functor Φ (see, for example, [L3]).

Therefore, if Φ is an isomorphism of categories then $\sigma_{\mathcal{Q}}$ is a braiding or a symmetry in the category $\mathcal{B}$.

2. Algebraic structures.

Let A be an algebra in a monoidal category $\mathcal{A}$ with multiplication law: $\mu : A \otimes A \longrightarrow A$.

Define a *quantization* $A_{\mathcal{Q}}$ of the algebra structure as the object $A_{\mathcal{Q}} = \Phi(A)$ of the category $\mathcal{B}$ with new product

$$\mu_{\mathcal{Q}} = \Phi(\mu) \circ \mathcal{Q}_{A,A} : \Phi(A)\hat{\otimes}\Phi(A) \longrightarrow \Phi(A).$$

It follows from the coherence conditions that the pair $(\Phi(A), \mu_{\mathcal{Q}})$ determines an algebra structure in the category $\mathcal{B}$.

Indeed, the naturality of $\mathcal{Q}$ gives us the following commutative diagram

$$\begin{array}{ccc} \Phi(A)\hat{\otimes}\Phi(A \otimes A) & \xrightarrow{id\hat{\otimes}\Phi(\mu)} & \Phi(A)\hat{\otimes}\Phi(A) \\ {\scriptstyle \mathcal{Q}_{A,A\otimes A}}\big\downarrow & & {\scriptstyle \mathcal{Q}_{A,A}}\big\downarrow \\ \Phi(A \otimes (A \otimes A)) & \xrightarrow{\Phi(id\otimes\mu)} & \Phi(A \otimes A). \end{array}$$

Therefore, for the proof of the associativity law we have

$$\begin{aligned} \mu_{\mathcal{Q}} \circ (id\hat{\otimes}\mu_{\mathcal{Q}}) &= \Phi(\mu) \circ \mathcal{Q}_{A,A} \circ (id\hat{\otimes}\Phi(\mu)) \circ \mathcal{Q}_{A,A} \\ &= \Phi(\mu) \circ \Phi(id \otimes \mu) \circ \mathcal{Q}_{A,A\otimes A}(id\hat{\otimes}\mathcal{Q}_{A,A}). \end{aligned}$$

In the same way we get

$$\mu_{\mathcal{Q}} \circ (\mu_{\mathcal{Q}}\hat{\otimes}id) = \Phi(\mu) \circ \Phi(id \otimes \mu) \circ \mathcal{Q}_{A\otimes A,A} \circ (\mathcal{Q}_{A,A}\hat{\otimes}id).$$

Analogously, if A is a coalgebra in the category $\mathcal{A}$ with diagonal $\Delta : A \longrightarrow A \otimes A$ then $A_{\mathcal{Q}} = \Phi(A)$ is a coalgebra in the category $\mathcal{B}$ with the new diagonal $\Delta_{\mathcal{Q}} = \mathcal{Q}_{A,A}^{-1} \circ \Phi(\Delta)$.

Let σ be a braiding in the category $\mathcal{A}$ and let (A, μ) be an algebra in $\mathcal{A}$. We define an algebra structure in $A^{\otimes 2}$, $\mu_\sigma^2 : A^{\otimes 2} \otimes A^{\otimes 2} \longrightarrow A^{\otimes 2}$ from the following commutative diagram:

$$\begin{array}{ccc}
(A \otimes A) \otimes (A \otimes A) & \xrightarrow{\mu_\sigma^2} & A \otimes A \\
{\scriptstyle id \otimes \sigma_{A,A} \otimes id}\Big\downarrow & & \Big\| \\
(A \otimes A) \otimes (A \otimes A) & \xrightarrow{\mu \otimes \mu} & A \otimes A.
\end{array}$$

Lemma. *The morphism μ_σ^2 determines an associative algebra structure in $\mathcal{A}$.*

Definition. [Mj]. A *braided* or *σ–bialgebra* in a monoidal category $\mathcal{A}$ is an algebra (A, μ) and coalgebra (A, Δ) structures, such that $\Delta : A \longrightarrow A \otimes A$ is an algebra homomorphism, where the last algebra is considered with the structure μ_σ^2.

Theorem. *Let (A, μ, Δ) be a σ–bialgebra. Then $(A_\mathcal{Q}, \mu_\mathcal{Q}, \Delta_\mathcal{Q})$ is a $\sigma_\mathcal{Q}$–bialgebra for any quantization $\mathcal{Q}$.*

3. Module structures.

Let X be a left A–module in category $\mathcal{A}$ with multiplication $\mu^l : A \otimes X \longrightarrow X$. *A quantization* of the module is the object $X_\mathcal{Q} = \Phi(X)$ with the new product

$$\mu^l_\mathcal{Q} = \Phi(\mu) \circ \mathcal{Q}_{A,X} : \Phi(A) \otimes \Phi(X) \longrightarrow \Phi(X).$$

One can show as above that $\mu_\mathcal{Q}$ determines a left $A_\mathcal{Q}$–module structure in the category $\mathcal{B}$.

In the obvious way we can apply the same procedure to right and bi–modules structures.

3.4. Let B be a bialgebra over k. Quantizations of the monoidal category $B-mod$ are determined by invertible elements $q \in B \otimes B$, such that the following conditions

$$(\Delta \otimes id)(q) \cdot (q \otimes 1) = (id \otimes \Delta)(q) \cdot (1 \otimes q), \tag{1}$$
$$(\varepsilon \otimes id)(q) = (id \otimes \varepsilon)(q) = 1, \tag{2}$$
$$q \cdot \Delta(b) = \Delta(b) \cdot q, \forall b \in B, \tag{3}$$

hold [L4].

Moreover, the action $\mathcal{Q}_{X,Y} : X \otimes Y \longrightarrow X \otimes Y$ is given by the formula

$$\mathcal{Q}_{X,Y}(x \otimes y) = q \cdot (x \otimes y). \tag{4}$$

Analogously one can describe quantizations of the forgetful functor $\Phi : B-mod \longrightarrow k - mod$. Any such quantization is given by an element $q \in B \otimes B$, but with conditions (1) and (2) only.

Definition. An element $q \in B \otimes B$ is called a *quantizer* of the bialgebra B if conditions (1) and (2) hold.

Theorem. *Any braiding element $\sigma \in B \otimes B$ of the bialgebra B is a quantizer of the bialgebra. Moreover, $q = \sigma$ determines the quantization of the category $B - mod$.*

Proof. Using conditions 1.3.(1) and (2), we get

$$(\Delta \otimes id)(\sigma) \cdot (\sigma \otimes 1) = \sigma_{23}\sigma_{13}\sigma_{12},$$
$$(id \otimes \Delta)(\sigma) \cdot (1 \otimes \sigma) = \sigma_{12}\sigma_{13}\sigma_{23},$$

and therefore equation 3.4.(1) is a consequence of the Yang–Baxter equation. □

3.5. In this section we write down quantizations of the main structures (see 3.3. above) in terms of quantizers.

1. Braidings.

Theorem. *Let $\sigma \in B \otimes B$ be a braiding element and let $q \in B \otimes B$ be a quantizer of the category $B - mod$ (condition 3.4.(3)) holds). Then*

$$\sigma_q = q^{-1} \cdot \sigma \cdot \tau(q)$$

is a braiding element, too.

Corollary. *If σ_1 and σ_2 are braiding elements, then $\sigma_1^{-1} \cdot \sigma_2 \cdot \tau(\sigma_1)$ is a braiding element, too.*

2. Algebraic structures.

Let A be an algebra in the category $B - mod$ with multiplication $\mu : a_1 \otimes a_2 \mapsto a_1 \cdot a_2$.

The quantization of A is an algebra $A_q = A$ with new multiplication

$$a_1 \underset{q}{\cdot} a_2 = \sum q'(a_1) \cdot q''(a_2),$$

where $q = \sum q' \otimes q''$.

We should remark that A_q is a B–algebra if condition 3.4.(3) holds.

3. Module structures.

Let X be a left A–module in the category $B - mod$ with multiplication $\mu^l : a \otimes x \mapsto a \cdot x$. The quantization is a left A_q–module $X_q = X$ with new multiplication

$$a \underset{q}{\cdot} x = \sum q'(a) \cdot q''(x).$$

In the same way we quantize right A–modules:

$$x \underset{q}{\cdot} a = \sum q'(x) \cdot q''(a).$$

3.6. Denote by $\mathcal{Q}(B)$ the set of all quantizers in bialgebra B.

Lemma. *Let $U(B) \subset B$ be the group of units of the algebra B, such that $\varepsilon(b) = 1$, $\forall b \in U(B)$. Then the formula*

$$b \in U(B), q \in \mathcal{Q}(B) \mapsto b(q) = \Delta(b) \cdot q \cdot b^{-\otimes 2},$$

where $b^{-\otimes 2} = b^{-1} \otimes b^{-1}$, determines $U(B)$–action on $\mathcal{Q}(B)$.

Proof. We have

$$\begin{aligned}
&(\Delta \otimes id)(b(q)) \cdot (b(q) \otimes 1) = \\
&(\Delta \otimes id)\Delta(b)(\Delta \otimes id)(q)(\Delta b^{-1} \otimes b^{-1})(\Delta(b) \otimes 1)(q \otimes 1)(b^{-\otimes 2} \otimes 1) \\
&= (\Delta \otimes id)\Delta(b)\, (\Delta \otimes id)(q) \cdot (q \otimes 1)\, b^{-\otimes 3},
\end{aligned}$$

and

$$\begin{aligned}
&(id \otimes \Delta)(b(q)) \cdot (1 \otimes b(q)) = \\
&(id \otimes \Delta)\Delta(b)(id \otimes \Delta)(q)(b^{-1} \otimes \Delta b^{-1})(1 \otimes \Delta(b))(1 \otimes q)(1 \otimes b^{-\otimes 2}) \\
&= (id \otimes \Delta)\Delta(b)(id \otimes \Delta)(q)\, (1 \otimes q)\, b^{-\otimes 3}.
\end{aligned}$$

For conditions (2) we have

$$(\varepsilon \otimes id)b(q) = b\, (\varepsilon \otimes id)(q)\, \varepsilon(b)\, b^{-1} = \varepsilon(b) = 1.$$

$\square$

Remark that in the case of forgetful functor Φ any unit preserving natural transformation λ from 3.2. has the form: $\lambda_X(x) = b \cdot x, \forall x \in X$, and for some invertible element $b \in B$, such that $\varepsilon(b) = 1$.

Denote by

$$\mathcal{Q}H^2(B) = \mathcal{Q}(B)/U(B)$$

the space of $U(B)$–orbits.

This space will be called the *non–linear cohomology* of the bialgebra B.

To explain this definition let us consider a "linearization" of $\mathcal{Q}H^2(B)$.

To do this, we fix a quantizer q and describe the "tangent plane" $T_q\mathcal{Q}(B)$ at the point.

Let

$$q(t) = q + \sum_{i \geq 1} \alpha_i\, t^i$$

be a formal curve on $\mathcal{Q}(B)$, $\quad \alpha_i \in B \otimes B, \quad (\varepsilon \otimes id)(\alpha) = (id \otimes \varepsilon)(\alpha) = 0$.

Substitute this expression in 3.4.(1) and look at coefficients of t^k, $k > 0$.

We get the equations on α_i :

$$\partial_q(\alpha_k) = \sum_{i+j=k, i \geq j, j>0} [\alpha_i, \alpha_j], \tag{1}$$

where

$$[\alpha_i, \alpha_j] = (\Delta \otimes id)(\alpha_i)(\alpha_j \otimes 1) - (id \otimes \Delta)(\alpha_i)(\alpha_j \otimes 1)$$
$$+ (\Delta \otimes id)(\alpha_j)(\alpha_i \otimes 1) - (id \otimes \Delta)(\alpha_j)(1 \otimes \alpha_i), \tag{2}$$

and the operator $\partial_q : B^{\otimes 2} \longrightarrow B^{\otimes 3}$ acts as follows

$$\partial_q(\alpha) = (\Delta \otimes id)(q)(\alpha \otimes 1) + (\Delta \otimes id)(\alpha)(q \otimes 1)$$
$$- (id \otimes \Delta)(q)(1 \otimes \alpha) - (id \otimes \Delta)(\alpha)(1 \otimes q). \tag{3}$$

Therefore, we have

$$T_q \mathcal{Q}(B) = ker\, \partial_q \bigcap ker(id \otimes \varepsilon) \bigcap ker(\varepsilon \otimes id).$$

Any curve $b(t) = 1 + b\,t + ...$ on the group $U(B)$ determines a curve

$$q(t) = \Delta(b(t))\, q\, b(t)^{-\otimes 2} = q + t(\Delta(b)q - q(b \otimes 1 + 1 \otimes b)) + ...$$

on the space $\mathcal{Q}(B)$.

Note that $\varepsilon(b) = 0$.

Therefore, tangent vectors to the orbit $U(B)q$ are elements of $Im\, \partial_q \bigcap ker\, \varepsilon$, where $\partial_q : B \longrightarrow B^{\otimes 2}$ is the following operator

$$\partial_q(b) = \Delta(b)q - q(b \otimes 1 + 1 \otimes b).$$

It follows from the above constructions that the sequence

$$B \xrightarrow{\partial_q} B^{\otimes 2} \xrightarrow{\partial_q} B^{\otimes 3}$$

is a complex, and linearization of non–linear cohomology space may be identified with the second cohomology group $H^2(B, q)$ of the normalized complex.

We should note also that the complex above coincides with the standard complex for Hochschild cohomology of the coalgebra at the point $q = 1$.

Let $\alpha \in B \otimes B$ be a tangent vector to $\mathcal{Q}(B)$, i.e.

$$\partial_q(\alpha) = 0, \qquad \text{and} \qquad (\varepsilon \otimes id)(\alpha) = (id \otimes \varepsilon)(\alpha) = 0.$$

To build up a curve 3.5. $q(t)$, such that $\alpha_1 = \alpha$ we need some additional conditions on α.

Namely, take $k = 2$ in formula 3.5.(1).

We get

$$\partial_q(\alpha_2) = [\alpha, \alpha]. \tag{1}$$

It is easy to check that the condition

$$[\alpha, \alpha] = 0\, mod\, Im\, \partial_q$$

depends on the cohomology class of α only, and $[\alpha, \alpha] \in Ker\, \partial_q$, for all $\alpha \in ker\, \partial_q$. Therefore, we obtain a bracket

$$[,]_q : H^2(B, q) \otimes H^2(B, q) \longrightarrow B^{\otimes 3}/Im\, \partial_q.$$

This bracket will be called a *q–bracket.*

Hence, a vector $\bar{\alpha} \in T_q \mathcal{Q}H^2(B)$ is a tangent vector to some curve on $\mathcal{Q}H^2(B)$ if $[\bar{\alpha}, \bar{\alpha}]_q = 0$.

Definition. A *q–Poisson structure* on the bialgebra B is a second cohomology class $\bar{\alpha} \in H^2(B, q)$, such that

$$[\bar{\alpha}, \bar{\alpha}]_q = 0.$$

Examples.

(1) Let us consider the category of G–graded modules over an Abelian finite group G. Then conditions 3.4. mean that any quantizer q is a 2–cocycle on the group with values in the unit group $U(k)$: $\quad q \in Z^2(G; U(k))$.

Therefore, in the given case : $\mathcal{Q}H^2(k(G)) = H^2(G, U(k))$.

(2) Let B be a commutative and co-commutative bialgebra over $k = \mathbb{C}$. We consider quantizers of the *Moyal type:*

$$q = exp(\alpha),$$

for some element $\alpha \in B \otimes B$.

For this case the main equation 3.4.(1) takes the form:

$$\alpha \otimes 1 - (id \otimes \Delta)(\alpha) + (\Delta \otimes id)(\alpha) - 1 \otimes \alpha = 0.$$

The last equation means that α isa 2-cocycle on the *coalgebra* B with values in $\mathbb{C}$.

For instance, if $B = \mathcal{D}(\mathbb{T}^n)$ is the Hopf algebra of distributions on n–dimentional torus $\mathbb{T}^n$, and α is an invariant Poisson structure on $\mathbb{T}^n$, then the above formula gives the Moyal quantization [BFFLS,V].

(3) Let $\imath : G_1 \longrightarrow G$ be a subgroup of group G. Then $\imath$ determines a Hopf algebra homomorphism $\imath_* : k[G_1] \longrightarrow k[G]$ (or $\imath_* : \mathcal{D}(G_1) \longrightarrow \mathcal{D}(G)$ for the case of compact Lie groups), and $\imath_*(q)$ is a quantizer on G, if q is a quantizer on G_1.

Applying the remark to a maximal torus of a Lie group G, we obtain a class of Moyal type quantizations [cf.R].

3.7. Let G be a semi–simple Lie group, $k = \mathbb{C}$, and $\mathcal{C}$ a monoidal category of finite dimensional G–modules over $\mathbb{C}$.

Denote by $K(G)$ the representation algebra of finite dimensional G–modules over $\mathbb{C}$.

In this section we show that quantizations of $\mathcal{C}$ may be described in terms of multiplicative 2–cohomologies of $K(G)$.

To do this, we introduce a new algebra $E(\hat{G})$ generated by all formal finite sums $f = \sum_{\gamma \in \hat{G}} f(\gamma)\,\gamma$, where $\hat{G}$ is the set of all the finite dimensional irreducible representations of G, and $f(\gamma) \in Hom_G(nX_\gamma, nX_\gamma) \simeq Mat_n(\mathbb{C})$, for some natural $n \in \mathbb{N}$.

Here we denote by X_γ a representative module of γ.

We convert $E(\hat{G})$ into an algebra over $K(G)$ by introducing the following operations:

$$f + g \stackrel{\text{def}}{=} \sum_{\gamma \in \hat{G}} (f(\gamma) \oplus g(\gamma))\,\gamma,$$

and

$$f \cdot g \stackrel{\text{def}}{=} \sum_{\gamma \in \hat{G}} (f(\gamma) \otimes g(\gamma))\,\gamma,$$

with $K(G)$–action:

$$\alpha \cdot f \stackrel{\text{def}}{=} \sum_{\gamma \in \hat{G}} f(\gamma)\, \alpha\gamma = \sum_{\delta, \gamma \in \hat{G}} (id \otimes f(\gamma))|_\delta\, \delta,$$

where $\alpha\gamma = \alpha \otimes \gamma = \sum_{\delta \in \hat{G}} m^\delta_{\alpha\gamma} \delta$, and $|_\delta$ is the restriction on the δ–component, $m^\delta_{\alpha\gamma} \in \mathbb{N}$ are the multiplicities of δ in the tensor product $\alpha \otimes \gamma$.

Any quantization $\mathcal{Q}$ of the monoidal category determines morphisms

$$\omega_{\alpha,\beta} = \mathcal{Q}_{X_\alpha, X_\beta} : X_\alpha \otimes X_\beta \longrightarrow X_\alpha \otimes X_\beta$$

for all $\alpha, \beta \in K(G)$.

We write down the morphisms in the form

$$\omega_{\alpha,\beta} = \sum_{\gamma \in \hat{G}} \omega_{\alpha,\beta}(\gamma)\, \gamma$$

where $\omega_{\alpha,\beta}(\gamma)$ is the restriction of $\omega_{\alpha,\beta}$ on the γ– component.

Therefore, any quantization $\mathcal{Q}$ determines 2–cochain

$$\omega : K(G) \times K(G) \longrightarrow E(\hat{G}),$$

with the additional condition

$$\omega_{\alpha,\beta}(\gamma) \in Hom_G(n^\gamma_{\alpha,\beta} X_\gamma, n^\gamma_{\alpha,\beta} X_\gamma) \simeq Mat_{n^\gamma_{\alpha,\beta}}(\mathbb{C}), \tag{1}$$

if $\alpha\beta = \sum_{\gamma \in \hat{G}} n^\gamma_{\alpha,\beta}\, \gamma$.

Moreover, ω is a normalized 2-cochain : $\omega_{\alpha,1} = \omega_{1,\alpha} = 1$, and commutativity of diagram 3.2. yields the following multiplicative 2-cocycle property

$$\omega_{\alpha,\beta\gamma} \cdot \alpha(\omega_{\beta,\gamma}) = \omega_{\alpha\beta,\gamma} \cdot \gamma(\omega_{\alpha,\beta}). \tag{2}$$

We say that ω is a *multiplicative 2-cocycle* if (2) holds for all $\alpha, \beta, \gamma \in K(G)$.

We say that ω is a *restricted 2-cocycle* if ω is a multiplicative 2-cocycle satisfying the additional condition (1).

Summarizing, we obtain the following

Theorem. *Any quantization of the monoidal category of finite dimensional G–modules is given by a multiplicative normalized and restricted 2-cocycle on the representation algebra with values in $E(\hat{G})$.*

Remark. Let G be a compact Lie group. Then in the same way one can describe quantizations of the monoidal category of unitary modules.

3.8. In this section we consider quantizations of the category such that $\omega_{\alpha,\beta}(\gamma) \in \mathbb{C}^*$, for all $\alpha, \beta, \gamma \in \hat{G}$, and

$$\omega(\alpha, \beta) = exp(2\pi\imath\, \theta(\alpha, \beta)), \tag{1}$$

for some 2-cochain $\theta : K(G) \times K(G) \longrightarrow E(\hat{G})$, where $\theta_{\alpha,\beta}(\gamma) \in \mathbb{C}$.

Let
$$d : C^k(K(G), E(\hat{G})) \longrightarrow C^{k+1}(K(G), E(\hat{G}))$$
be the Hochschild differential .

Then $\omega_{\alpha,\beta}$ commute and therefore conditions 3.7.(2) take the form

$$d(\theta)(\alpha,\beta,\gamma) = \alpha(\theta(\beta,\gamma)) - \theta(\alpha\beta,\gamma) + \theta(\alpha,\beta\gamma) - \gamma(\theta(\alpha,\beta)) \in \mathbb{Z}. \qquad (2)$$

Denote by $C^k_{\mathbb{Z}}(K(G), E(\hat{G})) \subset C^k(K(G), E(\hat{G}))$ the integer subcomplex and let $C^k_{\mathbb{C}/\mathbb{Z}}(K(G), E(\hat{G}))$ be the quotient complex.

Then condition (2) means that θ is a 2-cocycle in the quotient complex.

Therefore, we may reformulate theorem 3.7. in the following way:

Theorem. *Quantizations of type (1) are determined by Hochschild 2-cocycles of the quotient complex* $C^{\bullet}_{\mathbb{C}/\mathbb{Z}}(K(G), E(\hat{G}))$.

Examples.

(1) Let us consider the category of G-modules over an Abelian finite group G, $k = \mathbb{C}$.

Let $\hat{G}$ be the dual group. Then we have $X_\alpha \otimes X_\beta = X_{\alpha\beta}$, fot all $\alpha, \beta \in \hat{G}$.

Therefore, $\omega_{\alpha,\beta} = \tilde{\omega}_{\alpha,\beta} \cdot (\alpha\beta)$, where

$$\tilde{\omega} : \hat{G} \times \hat{G} \longrightarrow \mathbb{C}^*$$

is a 2-cocycle.

(2) Applying the same construction to the category of finite dimentional $\mathbb{T}^n$–modules, we get the Hochschild 2-cocycle:

$$\tilde{\omega} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{C}^*.$$

(3) The same construction may be applyed for the case of arbitrary compact Lie group G, if we consider the "1-particle" interaction: $\omega_{\alpha,\beta} = \tilde{\omega}_{\alpha\bullet\beta}(\alpha \bullet \beta)$, where $\alpha \bullet \beta$ is the highest part of the representation $\alpha\beta$.

(4) Let $k = \mathbb{C}$ and $G = S_3$ be the permutation group on three elements. The representation algebra $k(S_3)$ is generated by x_1–non-trivial 1-dimentional irreducible represantation, and x_2–2-dimentional irreducible representation with the following relations:
$$x_1^2 = 1, \quad x_1x_2 = x_2, \quad x_2^2 = x_2 + x_1 + 1.$$

Set $\omega_{ij} = \omega_{x_ix_j}$.

It easy to find that

$$\omega_{11} = a^2 \cdot 1, \quad \omega_{12} = \omega_{21} = a \cdot x_2, \quad \omega_{22} = ab \cdot 1 + b \cdot x_1 + c \cdot x_2,$$

for some $a, b, c \in \mathbb{C}^*$.

We have the following action of element $h = h_1 \cdot x_1 + h_2 \cdot x_2$ on ω:

$$h(\omega_{11}) = h_1^{-2}\omega_{11}, \quad h(\omega_{12}) = h_1^{-1}\omega_{12},$$

and

$$h(\omega_{22}) = h_2^{-1}c \cdot x_2 + h_1 h_2^{-2} b \cdot x_1 + h_2^{-2} ab \cdot 1.$$

Therefore, we have 1-parameter family of quantizations, considered up to trivial.

3.9. Let $\mathcal{C}$ be a monoidal category $B - mod$ for some k–bialgebra B and let $\mathcal{C}_0$ be a monoidal category of $k - mod$.

In this section we describe quantizations of the category $\mathcal{G}r(\mathcal{C})$ of $\mathbb{N}$– graded objects over $\mathcal{C}$ and quantizations of the forgetful functor $\Phi : \mathcal{G}r(\mathcal{C}) \longrightarrow \mathcal{G}r(\mathcal{C}_0)$.

Let $\hat{\mathcal{Q}}$ be a quantization of the forgetful functor. The quantization is given by the quantizer $\{q_{n,m}\}$, where

$$\hat{\mathcal{Q}}_{X,Y}(x_n \otimes y_m) = q_{n,m} \cdot (x_n \otimes y_m), \tag{0}$$

and $q_{n,m} \in B \otimes B, \quad n, m \in \mathbb{N}$.

Suppose that $q_{0,0} = q$ is a quantizer of the bialgebra.

Now equations 3.4.(1) and 3.4.(2) take the form:

$$(id \otimes \Delta)(q_{n,m+k}) \cdot (1 \otimes q_{m,k}) = (\Delta \otimes id)(q_{n+m,k}) \cdot (q_{n,m} \otimes 1), \tag{1}$$

$$(\varepsilon \otimes id)(q_{0,n}) = (id \otimes \varepsilon)(q_{n,0}) = 1. \tag{2}$$

Apply the morphism $id \otimes \varepsilon \otimes id$ to both sides of (1).

We get

$$q_{n,m+k} \cdot (1 \otimes p_{m,k}) = q_{n+m,k} \cdot (\bar{p}_{n,m} \otimes 1), \tag{3}$$

where $p_{m,k} = (\varepsilon \otimes id) q_{m,k}, \quad \bar{p}_{m,n} = (id \otimes \varepsilon) q_{n,m}$.

It follows from (2) that $p_{0,k} = 1, \quad \bar{p}_{n,0} = 1$.

Apply $\varepsilon \otimes id$ and $id \otimes \varepsilon$ to equation (3).

We get

$$p_{n,m+k} \cdot p_{m,k} = p_{m+n,k} \cdot \varepsilon(\bar{p}_{n,m}), \tag{4}$$

$$\bar{p}_{n,m+k} \cdot \varepsilon(p_{m,k}) = \bar{p}_{n+m,k} \cdot \bar{p}_{n,m}. \tag{5}$$

Let

$$f_{n,m} = \varepsilon(p_{n,m}) = \varepsilon(\bar{p}_{n,m}).$$

By applying counit ε to equations (4) or (5), we get

$$f_{n,m+k} \cdot f_{m,k} = f_{m+n,k} \cdot f_{n,m}.$$

It follows that $f_{n,m}$ determines a 2-cocycle on the group $\mathbb{Z}$ and therefore

$$f_{n,m} = \frac{f(n)f(m)}{f(n+m)}$$

for some function $f : \mathbb{Z} \longrightarrow U(k)$.

This means that up to equivalence we may suppose that $f_{n,m} \equiv 1$.

In the last case equations (4) and (5) take the form

$$\begin{cases} p_{n,m+k} \cdot p_{m,k} = p_{n+m,k}, & p_{0,k} = 1, \\ \bar{p}_{n+m,k} \cdot \bar{p}_{n,m} = \bar{p}_{n,m+k}, & \bar{p}_{n,0} = 1, \end{cases}$$

It follows from the system that

$$p_{n,m} = g(n+m-1)g(m-1)^{-1},$$

for some function $g : \mathbb{Z} \longrightarrow A$, and analogously

$$\bar{h}_{n,m} = h(n+m-1)h(n-1)^{-1},$$

for some function $h : \mathbb{Z} \longrightarrow A$.

Substitute this expressions in equation (3).

We get $h \equiv 1, \quad g \equiv 1$ and $q_{n.m} = Q(n+m-1)$, for some function $Q : \mathbb{Z} \longrightarrow A \otimes A$.

Now it follows from equation (1) that Q is a constant function.

Summarizing, we obtain the following

Theorem. *Any quantization $\hat{\mathcal{Q}}$ of the forgetful functor $\Phi : \mathcal{G}r(B - mod) \longrightarrow \mathcal{G}r(k - mod)$ is given by formula (0), where*

$$q_{n,m} = \frac{f(n)f(m)}{f(n+m)} \cdot q$$

for some function $f : \mathbb{Z} \longrightarrow U(k)$ and quantizer q.

Any two quantizations with given quantizer q are equivalent.

The proof of the theorem shows that the same result also holds for quantizations of the category.

3.10. In this section we apply the above quantization procedure to the modules of internal homomorphisms in the monoidal category $H - mod$, where H is a Hopf algebra.

To do this, we remark that the composition

$$f \in Hom(Y, Z),\, g \in Hom(X, Y) \mapsto f \circ g \in Hom(X, Z)$$

defines an associative partially determined product in the totality of all internal homomorphisms.

Moreover, we shall identify elements of modules $x \in X$ with internal homomorphisms $x : k \longrightarrow X$, where $x : 1 \mapsto x$.

In terms of this identification, the evaluation map

$$x \in X,\, f \in Hom(X, Y) \mapsto y = f(x) \in Y$$

is the product $f \circ x$.

Let $\mathcal{Q}$ be a quantization of the category, given by the quantizer $q \in H \otimes H$.

We define a new composition law $f \underset{q}{\circ} g$ as above

$$f \underset{q}{\circ} g = \sum q'(f) \circ q''(g).$$

For evaluation morphism we have

$$f_q(x) := f \underset{q}{\circ} x = \sum q'(f)(q''(x)) = \sum q'_{(1)}(f(Sq'_{(2)})q''(x)).$$

Note that $f \underset{q}{\circ} g = f \circ g$ if f or g is a morphism of the category.

As above new composition defines an associative partially determined product on the totality of all the internal homomorphisms.

3.11. Here we apply the above procedure of quantizations of internal homomorphisms to modules of braided differential operators.

We start with the bimodule case. Let σ be a braiding and A a σ– commutative algebra in the category. Let X be an $A - A$ bimodule.

Let δ^r_a and δ^l_a be the δ–operations in the bimodule X, and

$$\delta^l_{q,a}(x) = a \underset{q}{\cdot} x - \sum \sigma'_q(x) \underset{q}{\cdot} \sigma''_q(a),$$
$$\delta^r_{q,a}(x) = x \underset{q}{\cdot} a - \sum \sigma'_q(a) \underset{q}{\cdot} \sigma''_q(x),$$

δ–operations in the $A_q - A_q$–bimodule X_q.

Lemma. *One has*

$$\delta^l_{q,a}(x) = \sum \delta^l_{q'(a)}(q''(x)),$$

and

$$\delta^r_{q,a}(x) = \sum \delta^r_{q''(a)}(q'(x)).$$

Proof. We prove, for example, the first equality. One has

$$\begin{aligned}
& a \underset{q}{\cdot} x - \sum \sigma'_q(x) \underset{q}{\cdot} \sigma''_q(a) = \\
& \sum q'(a) \cdot q''(x) - \sum q'\sigma'_q(x) \cdot q''\sigma''_q(a) = \\
& \sum q'(a) \cdot q''(x) - \sum \sigma' q''(x) \cdot \sigma'' q'(a) = \\
& \sum \delta^l_{q'(a)}(q''(x)).
\end{aligned}$$

□

Corollary. *There exist embeddings* $X_\sigma \subset (X_q)_{\sigma_q}$.

Applying the result to modules of braided differential operators, we obtain the following

Theorem. *For any braided σ–differential operator $\nabla \in Diff_*^\sigma(X,Y)$ internal homomorphism ∇_q, where $\nabla_q(x) \stackrel{\text{def}}{=} \nabla \underset{q}{\circ} x$, is a braided σ_q–differential operator.*

The correspondence $\nabla \mapsto \nabla_q$ determines morphisms

$$\hat{q} : Diff_*^\sigma(X,Y) \longrightarrow Diff_*^{\sigma_q}(X_q, Y_q)$$

of modules of braided differential operators.

The morphism preserves the composition $\nabla_1 \circ \nabla_2 \mapsto (\nabla_1)_q \underset{q}{\circ} (\nabla_2)_q$ and the order of braided differential operators.

Definition. The operator $\nabla_q \in Diff_i^{\sigma_q}(X_q, Y_q)$ will be called a *quantization* of the operator $\nabla \in Diff_i^\sigma(X,Y)$.

3.12. Let $\nabla \in Der(A,X)$ be a σ–derivation of algebra A with values in a left A–module X.

Then $\nabla_q(1) = \nabla(1) = 0$, and therefore ∇_q is a derivation: $\nabla_q \in Der(A_q, X_q)$.

Applying the quantization to the representative objects, we obtain a morphism of braided differential forms

$$\hat{q} : \Omega^1(A) \longrightarrow \Omega^1(A_q).$$

Note that H–invariance of the differential d gives us $d_q = d$, and therefore morphism $\hat{q}$ has the form:

$$\hat{q} : a\, db \in \Omega^1(A) \mapsto a \underset{q}{\cdot} db \in \Omega^1(A_q).$$

Moreover,we can define in the same way morphisms

$$\hat{q}_\phi : \Omega^1(A,\phi) \longrightarrow \Omega^1(A_q,\phi),$$

where

$$\hat{q}_\phi : a\, d_\phi b \mapsto a \underset{q}{\cdot} d_\phi b,$$

for any differential prolongation of σ is given by the element ϕ.

In an obvious way the morphism may be extended to a homomorphism of $\hat{\sigma}$–commutative algebra $\Omega^*(A,\phi)$ into the $\hat{\sigma}_q$–commutative algebra $\Omega^*(A_q,\phi)$.

Theorem. *A quantization generates morphism of braided differential forms*

$$\hat{q}_\phi : \Omega^*(A,\phi) \longrightarrow \Omega^*(A_q,\phi)$$

such that

(1) *$\hat{q}_\phi$ is a morphism of a $\hat{\sigma}$–commutative algebra into a $\hat{\sigma}_q$–commutative algebra, considered with respect to $\underset{q}{\wedge}$ multiplication,*

(2) $\hat{q}_\phi \circ d_\phi = d_\phi \circ \hat{q}_\phi$.

Denote by $\Omega^*(A,\phi)_q = \sum_{i>0} \Omega^i(A,\phi)_q$ the kernel of $\hat{q}_\phi$. It is an ideal of $\hat{\sigma}$–commutative algebra closed with respect to d_ϕ.

A *Quantum cohomology kernel* $H_q^i(A,\phi)$ of the σ–commutative algebra A and a quantization q is the cohomology of the complex

$$\Omega^1(A,\phi)_q \xrightarrow{d_\phi} \Omega^2(A,\phi)_q \xrightarrow{d_\phi} \cdots \longrightarrow \Omega^i(A,\phi)_q \xrightarrow{d_\phi} \Omega^{i+1}(A,\phi)_q \longrightarrow \cdots.$$

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