# What do we need to build up Quantum Mechanics on Manifolds? The Lecture given at the Mathematical Institute of the Oslo University, 21.03.96

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**Abstract**

In this lecture we are analysing the relationships between quantum mechanics and theory of partial differential equations. We closely follow to the De Broglie and Schrödinger picture. Namely, we are considering the well known *wave–particle duality* as a relation between solutions of partial differential equations, describing waves, and singularities of solutions, that is particles. Our analysis of these relations shows that necessary ingredients of any Quantum Mechanical Picture are two connections. The first one is a in the tangent bundle of configuration manifold, and the second one is a connection in the trivial linear bundle.

We are considering also mechanical systems equipped with an inner structure and show that quantization of these systems require a linear connection in the corresponding vector bundle.

These are gravity and electromagnetic fields, or Yang-Mills field if the configuration space is the Minkowski space, and should be considered as a natural generalization of these fields for the case of general mechanical systems.

I'd like to stress two essential points in our approouch: (1) the described quantization procedure is an invariant under the diffeomorphism group and doesn't depend on choise of local coordinates, and (2) the procedure is a constructive one.

Explisit formulae for quantizations of some mechanical systems and the corresponding star-products are given.

# Contents

# Chapter 1

# Introduction

This paper is an expanded version of my talk given at Mathematical Institute of the Oslo University . We are analysing relationships between quantum mechanics (or more precisely, wave mechanics) and theory of partial differential equations. We closely follow to the De Broglie and Schrödinger original ideas. Namely, we are considering the well known wave–particle duality (this is the main idea of De Broglie) as a relation between solutions of partial differential equations (" waves" or wavefunctions) and singularities of solutions, "particles". In this case one could speak mathematically about "classical mechanics" associated with corresponding wave mechanics. We are analysing some generalizations of these relations for modules of differential operators acting in vector bundles. This case corresponds more sophisticated "mechanics", we call them *mechanical systems equipped with inner structure.* Quantum statistics and fractional statistics are the more important examples of such structures.

Hitherto there exists the following well known mathematical interpretations of Quantum Mechanics

1. Quantum Mechanics as a part of the Operator Algebras Theory,
2. Quantum Mechanics as a part of the Theory of Deformations of Algebras, so called Deformation Quantization.

The last approach was started by Flato, Fronsdal, Lichnerowicz in [1]. They introduced an important notion of the star-product in order to give an alternative formulation of quantum mechanics without operators.

The explicit star-products based on the classical Moyal quantization [7] were constructed for Lie groups [3] and some homogeneous spaces [1].

This approach really exploit the observation that linear part of a deformation of an algebra is Hochschild 2-cocycle, and for the case of smooth commutative algebra by passing to the Hochschild cohomologies one gets a Poisson structure [8]. This gives a very instructive explanation of the role of symplectic geometry in classical and quantum mechanics.

On the other side this observation is a source of misunderstanding. Namely, it proved that (for the smooth algebra case) any Poisson structure could be served as a tangent vector to some deformation of the algebra. But existence in mathematics is not existence in physics. The existence in physics should be realized by some physical agents or fields. Our ignorance of these fields is (from our point of view) a source of main troubles in the quantization picture.

The problem is a more distinctive for the case of mechanical systems on manifolds. To analyse the problem we have to go back to origins of quantum mechanics. We are starting with analysis of the de Broglie's point of view and its Schrödinger's realization.

Firstly, we interpret "waves of Matter" as indication to the algebra of differential operators. Secondly, we consider the interpretation of "particles as singularities" in terms of wave fronts of PDE solutions. This leads us to the notion of Hamiltonian of differential operator which differs from the well known symbol. The difference becomes to be very essential when we analyse Quantum mechanics as an "arrow" between Hamiltonians and Differential operators. Namely this difference gives the Schrödinger operator. In the case when we substitute Hamiltonians by symbols the procedure shall produce the heat equation.

After understanding of quantizations as arrows a construction of Quantum Mechanics becomes to be a problem of mathematical technique. So,we find all necessity ingredients and build up the quantization procedure for mechanical system with any configuration space.

It is worth to mention that we obtain description of "quantum algebra" in terms of algebra of differential operators. Therefore, the general problem of realizations or *representations* of the algebra becomes to be the problem of modules over the algebra of differential operators, or, in other words, the problem of investigation of partial differential equations. This gives a new (quantum) sight on differential equations.

# Chapter 2

# Algebra of Differential Operators

## 2.1 Symbols and Hamiltonians

Let $\mathcal{F} = C^{\infty}(M, \mathbb{C})$ be the algebra of smooth $\mathbb{C}$-valued functions on smooth manifold $M$ (say space–time), and let $\mathbf{Diff}_k(\mathcal{F})$ be the $\mathcal{F}$-module of scalar differential operators or operators acting in $\mathcal{F}$ which have an order $\leq k$.

Then

$$\mathcal{A} = \mathbf{Diff}_*(\mathcal{F}) = \bigcup_{k \geq 0} \mathbf{Diff}_k(\mathcal{F})$$

is an associative algebra with respect to the composition of operators.

This algebra is a filtered algebra

$$\mathcal{A} = \bigcup_{k \geq 0} \mathcal{A}_k, \ \ \mathcal{A}_k \subset \mathcal{A}_{k+1}, \ \mathcal{A}_k \cdot \mathcal{A}_l \subset \mathcal{A}_{k+l}$$

by the construction where $\mathcal{A}_k = \mathbf{Diff}_k(\mathcal{F})$, and a *weakly-noncommutative* algebra.

The last statement means that the graded algebra associated with $\mathcal{A}$

$$\mathbf{Gr}_*(\mathcal{A}) = \sum_{k \geq 0} \mathbf{Gr}_k(\mathcal{A}), \text{ where } \mathbf{Gr}_k(\mathcal{A}) = \mathbf{Diff}_k(\mathcal{F}) \diagup \mathbf{Diff}_{k-1}(\mathcal{F}),$$

is a commutative.

There are few definitions of differential operators that doesn't depend on a choice of local coordinates. Here we will use the definition based on action of differential operators on oscillating functions.

Namely, let's define a *Hamiltonian* of differential operator $\Delta : \mathcal{F} \longrightarrow \mathcal{F},\ \Delta \in \mathbf{Diff}_k(\mathcal{F})$ as follows.

Fix a scale $\hbar > 0$ of the frequency line and consider the expression $e^{-\frac{\imath}{\hbar}\omega S}\Delta\left(e^{\frac{\imath}{\hbar}\omega S}f\right)$ where $f, S \in \mathcal{F}$, and $\omega \in \mathbb{R}$, as a function in $\omega$.

One gets a polynomial in $\omega$ :

$$e^{-\frac{\imath}{\hbar}\omega S}\Delta\left(e^{\frac{\imath}{\hbar}\omega S}f\right) = \omega^k H(dS) f + \omega^{k-1} H_{k-1}(S)(f) + \cdots + H_0(S)(f)$$

where $H = H_\Delta$ depends on the differential $dS$ only and determines a homogeneous function of degree $k$ on the cotangent bundle $\tau^* : T^*M \longrightarrow M$.

We call $H = H_\Delta$ *Hamiltonian* of $\Delta$.

The correspondence $\Delta \longmapsto H_\Delta$ determines an isomorphism between $\mathbf{Gr}_k(\mathcal{A})$ and homogeneous polynomials (along fibres) of degree $k$ on $T^*(M)$.

The notion of symbol is a more widely distributed in the theory of differential operators.

Namely, the contiguity class

$$smbl_k(\Delta) = \Delta \operatorname{mod} \mathbf{Diff}_{k-1}(\mathcal{F}) \in \mathbf{Gr}_k(\mathcal{A})$$

is called a *symbol* of the operator.

It is worth to note that there is a difference between symbols and Hamiltonians, and this difference becomes very essential in Quantum Mechanics.

**Example 1** *Let $\Delta = \sum_{s=1}^n a_s(x)\,\partial_s$ where $\partial_s = \frac{\partial}{\partial x_s}$, and $(x_1, ..., x_n)$ are local coordinates on $M$, be a vector field. Then*

$$smbl_1(\Delta) = \sum_{s=1}^n a_s(x)\, p_s$$

*where $(x_1, ..., x_n; p_1, ..., p_n)$ are the natural local coordinates on $T^*M$, and*

$$H_\Delta = \sum_{s=1}^n a_s(x)\left(\frac{\imath}{\hbar}p_s\right).$$

*If $\Delta = \sum_{l \le s=1}^n a_{ls}(x)\,\partial_l\partial_s + \sum_{s=1}^n a_s(x)\,\partial_s + +a_0(x)$, is a second order differential operator then*

$$smbl_2(\Delta) = \sum_{l \le s=1}^n a_s(x)\, p_l p_s,$$

*but*

$$H_{\Delta}=-\frac{1}{\hbar^{2}}\sum_{l\leq s=1}^{n}a_{s}\left(x\right)p_{l}p_{s}.$$

*In general if*

$$\Delta=\sum_{|\sigma|\leq k}a_{\sigma}\left(x\right)\partial^{\sigma}$$

*is a differential operator of* $k$*th order where* $\sigma=(\sigma_1,...,\sigma_n)$ *, and* $|\sigma|=\sigma_1+\cdots+\sigma_n$*,then*

$$H_{\Delta}=\sum_{|\sigma|=k}a_{\sigma}\left(x\right)\left(\frac{\imath}{\hbar}p\right)^{\sigma}.$$

The following relation holds between symbols and Hamiltonians

$$H_{\Delta}=\left(\frac{\imath}{\hbar}\right)^{k}smbl_{k}\left(\Delta\right), \tag{2.1}$$

when $\Delta\in\mathbf{Diff}_k\left(\mathcal{F}\right)$.

## 2.2 Poisson Structures

Let $\mathbf{D}=\mathbf{D}\left(M\right)$ be a $\mathcal{F}-$module of smooth vector fields on $M$, and let

$$\mathbf{S}^{*}\left(\mathbf{D}\right)=\mathcal{F}\oplus\mathbf{D}\oplus\mathbf{S}^{2}\left(\mathbf{D}\right)\oplus\cdots\oplus\mathbf{S}^{k}\left(\mathbf{D}\right)\oplus\cdots$$

be the symmetric algebra of the module $\mathbf{D}$.

Here $\mathbf{S}^k\left(\mathbf{D}\right)$ is the $k$th symmetric power of $\mathbf{D}$, and $\bullet$ is the symmetric product in the algebra.

There are two important interpretations of elements of the algebra:

1. identifying vector fields with linear (along fibres) functions on $\mathbf{T}^*M$ one can consider elements of $\mathbf{S}^k\left(\mathbf{D}\right)$ as homogeneous polynomials on $\mathbf{T}^*M$ of degree $k$, and

2. due to isomorphism

$$\mathbf{smbl}_{k}:\mathbf{Diff}_{k}\left(\mathcal{F}\right)\diagup\mathbf{Diff}_{k-1}\left(\mathcal{F}\right)\overset{\sim}{\longrightarrow}\mathbf{S}^{k}\left(\mathbf{D}\right)$$

   one can identify elements of $\mathbf{S}^k\left(\mathbf{D}\right)$ with symbols of differential operators.

In a usual way we introduce a *Poisson structure* on $\mathbf{S}^*(\mathbf{D})$by

$$[\cdot,\cdot]:\mathbf{S}^*(D)\times\mathbf{S}^*(D)\longrightarrow\mathbf{S}^*(D),$$

by setting

$$[smbl_k(\Delta),smbl_l(\nabla)]=smbl_{k+l-1}([\Delta,\nabla])$$

where $\Delta\in\mathbf{Diff}_k(\mathcal{F})$, and $\nabla\in\mathbf{Diff}_l(\mathcal{F})$.

Hamiltonians produce a new Poisson structure. Namely, one defines a bracket on Hamiltonians as follows:

$$\{H_\Delta,H_\nabla\}_\hbar\stackrel{\mathrm{def}}{=}H_{[\Delta,\nabla]}. \tag{2.2}$$

Due to this definition and relation 2.1 we obtain

$$\{H_\Delta,H_\nabla\}_\hbar=\left(\frac{\imath}{\hbar}\right)^{k+l-1}[smbl_k(\Delta),smbl_l(\nabla)],$$

or

$$\{smbl_k(\Delta),smbl_l(\nabla)\}_\hbar=\frac{\hbar}{\imath}[smbl_k(\Delta),smbl_l(\nabla)]. \tag{2.3}$$

**Example 2** *In the natural local coordinates* $(x_1,...,x_n;p_1,...,p_n)$ *on* $\mathbf{T}^*M$ *one has*

$$[p_s,x_t]=\delta_{st},$$

*but*

$$\{p_s,x_t\}_\hbar=-\imath\hbar\delta_{st}.$$

# Chapter 3

# Quantizations of Classical Mechanics

It is well known that singularities of partial differential equations (say, in the sense of distributions ) propagate along to Hamiltonian vector field $X_H$ corresponding to the Hamiltonian.

This way *waves* $\Longrightarrow$*particles* one can interpret as a realization of the De Broglie's idea of relationship between wave mechanics and the classical one.

Here we will discuss the opposite way: *particles*$\Longrightarrow$ *waves*.

Namely this arrow could be considered as a *quantization* (on the mechanical level).

From the algebraic viewpoint this passing can be identified with arrow (= $\mathcal{F}$-linear morphism)

$$\mathbf{S}^*(\mathbf{D}) \xrightarrow{\mathbf{q}_*} \mathbf{Diff}_*(\mathcal{F})$$

which splits all exact sequences

$$\mathbf{0} \longrightarrow \mathbf{Diff}_{k-1}(\mathcal{F}) \longrightarrow \mathbf{Diff}_k(\mathcal{F}) \underset{\mathbf{Ham}}{\overset{\overset{\mathbf{q}_k}{\leftarrow}}{\longrightarrow}} \mathbf{S}^k(\mathbf{D}) \longrightarrow \mathbf{0},$$

where $\mathbf{Diff}_k(\mathcal{F}) \underset{\mathbf{Ham}}{\longrightarrow} \mathbf{S}^k(\mathbf{D})$ is the map that sending any operator $\Delta$ to the Hamiltonian $H_\Delta$.

At first, we will assume that $\mathbf{q}_0 = \mathrm{id}$.

Let's analyse a construction (or existence) of $\mathbf{q}$ for $k = 1, 2$.

Put $k = 1$.

One has the following exact sequence

$$\mathbf{0} \longrightarrow \mathbf{Diff}_0(\mathcal{F}) = \mathcal{F} \longrightarrow \mathbf{Diff}_1(\mathcal{F}) \underset{\mathbf{Ham}}{\overset{\mathbf{q}_1}{\rightleftarrows}} \mathbf{D} \longrightarrow \mathbf{0}$$

and $H_X = -\imath\hbar X$ , if $X \in \mathbf{D}$.

Therefore, $\mathbf{q}_1$ has to have the form

$$\mathbf{q}_1 : X \in \mathbf{D} \longmapsto -\imath\hbar X + \omega(X) \in \mathbf{Diff}_1(\mathcal{F})$$

for some differential 1-form $\omega \in \mathbf{\Omega}^1(M, \mathbb{C})$ .

Below we will see that $\omega$ should be viewed as a connection in the trivial linear bundle $\mathbf{1} : M \times \mathbb{C} \longrightarrow M$ .

In the case when $\omega$ is a real differential form we shall call $\omega$ *vector potential.*

**Remark 1** *If $M$ is the Minkowski space then $\omega$ might be really identified with the vector potential of an electromagnetic field, and our mechanics should be considered as the mechanics of charged particles. In the case of general configuration spaces $\omega$ should be considered as a* generalized electromagnetic field.

It is easy to check that the Heisenberg relations for this case take the form

$$\mathbf{q}_1[X,Y] = \frac{\imath}{\hbar}[\mathbf{q}_1(X), \mathbf{q}_1(Y)] + d\omega(X,Y), \tag{3.1}$$

or

$$[\mathbf{q}_1(X), \mathbf{q}_1(Y)] = \mathbf{q}_1(\{X,Y\}_\hbar) - \imath\hbar d\omega(X,Y)$$

where $X, Y \in D$, and

$$[\mathbf{q}_1(X), f] = -\imath\hbar X(f), \tag{3.2}$$

$$\text{or} \tag{3.3}$$

$$\mathbf{q}_0(\{X,f\}_\hbar) = [\mathbf{q}_1(X), \mathbf{q}_0(f)], \tag{3.4}$$

if $X \in \mathbf{D},\ f \in \mathcal{F}$.

It is worth to mention that 3.1 determines a new Lie algebra structure into $\mathbf{Diff}_1(\mathcal{F})$ .

Namely, the pairing

$$(\mathbf{q}_1(X), f) \times (\mathbf{q}_1(Y,g)) \longrightarrow (\mathbf{q}_1([X,Y]), \mathbf{q}_1(X)(g) - \mathbf{q}_1(Y)(f))$$

gives the *twisted* or *deformed* Lie algebra structure into the algebra of differential operators of the 1st order.

Let $k=2$.

Consider the next exact sequence

$$\mathbf{0}\longrightarrow\mathbf{Diff}_1(\mathcal{F})\longrightarrow\mathbf{Diff}_2(\mathcal{F})\underset{\mathbf{Ham}}{\longrightarrow}\mathbf{S}^2(\mathbf{D})\longrightarrow\mathbf{0}.$$

To define $\mathbf{q}_2:\mathbf{S}^2(\mathbf{D})\longrightarrow\mathbf{Diff}_2(\mathcal{F})$ we look at a supplementary map

$$\widetilde{\mathbf{q}}_2:\mathbf{D}\underset{\mathcal{F}}{\otimes}\mathbf{D}\longrightarrow\mathbf{Diff}_2(\mathcal{F})$$

which on decomposable tensors has a form

$$\widetilde{\mathbf{q}}_2(X\otimes Y)=\mathbf{q}_1(X)\circ\mathbf{q}_1(Y)+\imath\hbar\mathbf{q}_1(\Box_X(Y))+g(X,Y). \tag{3.5}$$

Here $X,Y\in\mathbf{D}$, $\circ$ is the composition of differential operators, and we denoted by $\Box_X(Y)\in\mathbf{D}$, and by $g(X,Y)\in\mathcal{F}$ corresponding components of the 1st order differential operator.

We obtain some properties of the hieroglyphic $\Box$ and $g$ by analysing the $\mathcal{F}$-linearity of the tensors $X\otimes Y$ and morphism $\mathbf{q}$.

Let's check $\mathcal{F}$-linearity in $X$. One has

$$\begin{aligned}&\widetilde{\mathbf{q}}_2(fX\otimes Y)=f\widetilde{\mathbf{q}}_2(X\otimes Y)\Longrightarrow\Box_{fX}(Y)=f\Box_X(Y),\\ \text{and}\qquad &g(fX,Y)=fg(X,Y);\end{aligned}$$

Respectively, for $\mathcal{F}$-linearity in $Y$, one has

$$\begin{aligned}\widetilde{\mathbf{q}}_2(X\otimes fY)&=\mathbf{q}_1(X)\circ\mathbf{q}_1(fY)+\imath\hbar\mathbf{q}_1(\Box_X(fY))+g(X,fY)=\\&\quad f\mathbf{q}_1(X)\circ\mathbf{q}_1(Y)-\imath\hbar X(f)\,\mathbf{q}_1(Y)+\imath\hbar\mathbf{q}_1(\Box_X(fY))\\&\quad+g(X,fY),\end{aligned}$$

and

$$f\widetilde{\mathbf{q}}_2(X\otimes Y)=f\mathbf{q}(X)\circ\mathbf{q}(Y)+\imath\hbar f\mathbf{q}_1(\Box_X(Y))+fg(X,Y).$$

Comparing the terms from $\mathbf{q}_1(\mathbf{D})$ and $\mathcal{F}$ in the decomposition $\mathbf{Diff}_1(\mathcal{F})\overset{\mathbf{q}_1}{\simeq}\mathbf{q}_1(\mathbf{D})\oplus\mathcal{F}$ one gets

$$\begin{aligned}\Box_X(fY)&=f\Box_X(Y)+X(f)\,Y,\\ g(X,f\cdot Y)&=f\cdot g(X,Y).\end{aligned} \tag{3.6}$$

In other words $\square$ is a linear connection in the tangent bundle $TM \longrightarrow M$ and $g$ is a covariant 2-tensor.

We define now $\mathbf{q}_2$ as follows:

$$\mathbf{q}_2 (X \bullet Y) = \frac{1}{2}\widetilde{\mathbf{q}}_2 (X \otimes Y + Y \otimes X) = \tag{3.7}$$

$$\frac{1}{2}(\mathbf{q}_1 (X) \circ \mathbf{q}_1 (Y) + \mathbf{q}_1 (Y) \circ \mathbf{q}_1 (X)) + \tag{3.8}$$

$$\frac{\imath\hbar}{2}\mathbf{q}_1 (\square_X (Y) + \square_Y (X)) + \frac{1}{2}(g (X, Y) + g (Y, X))$$

where

$$X \bullet Y = \frac{1}{2}(X \otimes Y + Y \otimes X)$$

is the symmetric product of vector fields.

Therefore, morphism $\mathbf{q}$ on the level $k = 2$ is determined by:

1. a connection $\square$ in the tangent bundle $\tau : \mathbf{T}M \longrightarrow M$, and

2. a symmetric covariant 2-tensor $\frac{1}{2}(g (X, Y) + g (Y, X)) \in \mathbf{S}^2 (\tau^*)$.

Moreover, by using these data one can define the quantization morphism $\mathbf{q}_2$ by the explicit formula

In other words to realize our quantum paradigm on the level of the 2nd order differential operators we need three components:

1. a linear connection $\omega$ in the trivial linear bundle $\mathbf{1} : M \times \mathbb{C} \longrightarrow M$, (or electromagnetic field in the case of the Minkowski space),

2. a linear connection $\square$ in the tangent bundle $\tau : \mathbf{T}M \longrightarrow M$,

   (or gravity in the case of the Minkowski space), and

3. a symmetric covariant 2-tensor .

In a similar way we'll build up morphisms

$$\mathbf{q}_k : \mathbf{S}^k (\mathbf{D}) \longrightarrow \mathbf{Diff}_k (\mathcal{F})$$

by the induction on $k$.

Firstly we define, as above, morphisms

$$\widetilde{\mathbf{q}}_k : \mathbf{D} \underset{\mathcal{F}}{\otimes} \mathbf{S}^{k-1} (\mathbf{D}) \longrightarrow \mathbf{Diff}_k (\mathcal{F})$$

by the formula

$$\widetilde{\mathbf{q}}_k(X\otimes Y)=\mathbf{q}_1(X)\circ\mathbf{q}_{k-1}(Y)+\imath\hbar\mathbf{q}_{k-1}\left(\square_X^{(k-1)}(Y)\right)+g_{k-2}(X,Y) \quad (3.9)$$

where $X\in\mathbf{D}$, $Y\in\mathbf{S}^{k-1}(\mathbf{D})$, and $\square_X^{(k-1)}(Y)\in\mathbf{S}^{k-1}(\mathbf{D})$, $g_{k-2}(X,Y)\in\mathbf{Diff}_{k-2}(\mathcal{F})$.

As above we see that hieroglyph $\square_X^{(k-1)}(Y)$ is really a covariant derivative of symmetric $(k-1)$-tensor $Y$ along vector field $X$, and therefore one can take for $\square^{(k-1)}$ to be the connection into $\mathbf{S}^{k-1}(\mathbf{D})$ generating by $\square$.

Now we'll define morphism $\mathbf{q}_k$ by the formula:

$$\mathbf{q}_k(X_1\bullet\cdots\bullet X_k)=\frac{1}{k}\sum_{a=1}^{k}\widetilde{\mathbf{q}}_k\left(X_a\otimes X_{(a)}\right)$$

where $X_1,...,X_k\in\mathbf{D}$, and $X_{(a)}=X_1\bullet\cdots\bullet X_{a-1}\bullet X_{a+1}\bullet\cdots\bullet X_k$.

Using 3.9 we obtain the explicit formula for $\mathbf{q}_k$ :

$$\mathbf{q}_k(X_1\bullet\cdots\bullet X_k)=\frac{1}{k}\sum_{a=1}^{k}\mathbf{q}_1(X_a)\circ\mathbf{q}_{k-1}\left(X_{(a)}\right)+ \quad (3.10)$$

$$\frac{\imath\hbar}{k}\mathbf{q}_{k-1}\left(\sum_{a=1}^{k}\sum_{b\neq a}\square_{X_a}(X_b)\left(X_{(a)}\right)_{(b)}\right)+g_{k-2}(X_1,...,X_k)$$

where $g_{k-2}:\mathbf{S}^k(\mathbf{D})\longrightarrow\mathbf{Diff}_{k-2}(\mathcal{F})$ is a covariant symmetric $k-$tensor with values into $\mathbf{Diff}_{k-2}(\mathcal{F})$.

Therefore, any quantization $\mathbf{q}$ is determined , up to arbitrary tensors $g_{k-2}$, $k=2,3,...,$by the two connections: the first one is a connection into the trivial linear bundle, and the second one is a connection into the tangent bundle.

We can rewrite formula (3.10) in the more instructive form.

Namely, let

$$\delta:\mathbf{S}^k(\mathbf{D})\longrightarrow\mathbf{D}\underset{\mathcal{F}}{\otimes}\mathbf{S}^{k-1}(\mathbf{D})$$

be the Spencer $\delta$-operator.

That is the operator that acts on the decomposable symmetric tensors as follows

$$\delta(X_1\bullet\cdots\bullet X_k)=\sum_{a=1}^{k}X_a\otimes X_1\bullet\cdots\bullet X_{a-1}\bullet X_{a+1}\bullet\cdots\bullet X_k.$$

We consider also a covariant differential

$$d_{\square}:\mathbf{D}\underset{\mathbb{C}}{\otimes}\mathbf{S}^{k-1}\left(\mathbf{D}\right)\longrightarrow\mathbf{S}^{k-1}\left(\mathbf{D}\right)$$

that acts in the usual way

$$d_{\square}\left(X\underset{\mathbb{C}}{\otimes}P\right)=\square_{X}\left(P\right).$$

In this notations formula (3.10) takes the form

$$k\cdot\mathbf{q}_{k}=\left(\mu\circ\left(\mathbf{q}_{1}\otimes\mathbf{q}_{k-1}\right)+\imath\hbar\mathbf{q}_{k-1}\circ d_{\square}\right)\circ\delta \tag{3.11}$$

where we denoted by $\mu$ the multiplication map

$$\mu:\mathbf{Diff}_{1}\left(\mathcal{F}\right)\underset{\mathbb{C}}{\otimes}\mathbf{Diff}_{k-1}\left(\mathcal{F}\right)\longrightarrow\mathbf{Diff}_{k}\left(\mathcal{F}\right).$$

**Local coordinates.** Let

$$\square_{\partial_{a}}\left(\partial_{b}\right)=\sum_{c=1}^{n}\Gamma_{ab}^{c}\left(x\right)\partial_{c}$$

and

$$\omega=\sum_{a=1}^{n}A_{a}\left(x\right)dx_{a}.$$

We have

$$\mathbf{q}_{1}\left(\partial_{a}\right)=\widehat{p_{a}}=-\imath\hbar\partial_{a}+A_{a}\left(x\right),$$

and

$$\begin{aligned}&\mathbf{q}_{2}\left(\partial_{a}\bullet\partial_{b}\right)=\widehat{p_{a}p_{b}}=\\&-\hbar^{2}\partial_{a}\circ\partial_{b}-\imath\hbar\left(\left(A_{a}\partial_{b}+A_{b}\partial_{a}\right)\right)-\frac{\imath\hbar}{2}\left(\partial_{a}\left(A_{b}\right)+\partial_{b}\left(A_{a}\right)\right)+A_{a}A_{b}+\\&\frac{h^{2}}{2}\sum_{c=1}^{n}\left(\Gamma_{ab}^{c}+\Gamma_{ba}^{c}\right)\partial_{c}+\frac{\imath\hbar}{2}\sum_{c=1}^{n}\left(\Gamma_{ab}^{c}+\Gamma_{ba}^{c}\right)A_{c}+g_{ab}\left(x\right).\end{aligned}$$

**Example 3** *Consider a Hamiltonian*

$$H=-p_{0}+\frac{1}{2m}\sum_{i=1}^{n}p_{i}^{2}+U\left(x\right).$$

*Then*

$$\widehat{H} = -\mathbf{q}_1(\partial_0) + \frac{1}{2m}\sum_{i=1}^{n}\mathbf{q}_2\left(\partial_i^2\right) + U(x) = \tag{3.12}$$
$$-\left(\frac{\hbar}{\imath}\partial_0 + A_0\right) + \frac{1}{2m}\sum_{a=1}^{n}\left(\frac{\hbar}{\imath}\partial_a + A_a\right)^2 +$$
$$\frac{\imath\hbar}{2}\sum_{a=1}^{n}\sum_{b=1}^{n}\Gamma_{aa}^{b}(x)\left(\frac{\hbar}{\imath}\partial_b + A_b\right) + U(x) + \frac{1}{2m}\sum_{i=1}^{n} g_{ii}(x).$$

## 3.1 The star products

Any quantization

$$\mathbf{q}_*\colon \mathbf{S}^*(\mathbf{D}) \longrightarrow \mathbf{Diff}_*(\mathcal{F})$$

determines a new product into the algebra $\mathbf{S}^*(\mathbf{D})$.

Namely, let's define *the star-product* as follows

$$a \underset{\mathbf{q}}{\bigstar} b = \mathbf{q}_*^{-1}\left(\mathbf{q}_*(a) \circ \mathbf{q}_*(b)\right) \tag{3.13}$$

where $a, b \in \mathbf{S}^*(\mathbf{D})$.

**Theorem 1** *1. $\mathbf{S}^*(\mathbf{D})$ is an associative algebra with respect to star-product 3.13.*

*2. In the algebra we have the following relations*

$$f \underset{\mathbf{q}}{\bigstar} X = fX, \quad X \underset{\mathbf{q}}{\bigstar} f = fX - \imath\hbar X(f) \tag{3.14}$$

$$X \underset{\mathbf{q}}{\bigstar} Y = X \bullet Y - \frac{\imath\hbar}{2}\left([X,Y] + \square_X(Y) + \square_Y(X) + d\omega(X,Y)\right)$$
$$-g(X,Y) \tag{3.15}$$

*where $X, Y \in \mathbf{D}$, and $f \in \mathcal{F}$.*

**Proof.** To prove the associativity one has

$$\left(a \underset{\mathbf{q}}{\bigstar} b\right) \underset{\mathbf{q}}{\bigstar} c = \mathbf{q}_*^{-1}\left(\mathbf{q}_*\left(a \underset{\mathbf{q}}{\bigstar} b\right) \circ \mathbf{q}_*(c)\right) = \mathbf{q}_*^{-1}\left(\left(\mathbf{q}_*(a) \circ \mathbf{q}_*(b)\right) \circ \mathbf{q}_*(c)\right)$$

and

$$a \underset{\mathbf{q}}{\bigstar} \left( b \underset{\mathbf{q}}{\bigstar} c \right) = \mathbf{q}_*^{-1} \left( \mathbf{q}_* (a) \circ \mathbf{q}_* \left( b \underset{\mathbf{q}}{\bigstar} c \right) \right) = \mathbf{q}_*^{-1} \left( \mathbf{q}_* (a) \circ \left( \mathbf{q}_* (b) \circ \mathbf{q}_* (c) \right) \right).$$

Therefore, the associativity law for the star-product is a result of associativity of the algebra $\mathbf{Diif}_* (\mathcal{F})$ and the invertibility of $\mathbf{q}_*$.

The first relations in 3.14 are trivial consequence of the definition of $\mathbf{q}_1$. To prove the second part of the relations we have:

$$\mathbf{q}_1 (X) \circ \mathbf{q}_1 (Y) = -\hbar^2 X \circ Y - \imath\hbar \left( X (\omega (Y)) + \omega (X) \cdot Y + \omega (Y) \cdot X \right) + \omega (X) \cdot \omega (Y)$$

and

$$\begin{aligned} &\mathbf{q}_2 (X \bullet Y) = -\frac{\hbar^2}{2} \left( X \circ Y + Y \circ X - \Box_X (Y) - \Box_Y (X) \right) - \\ &\frac{\imath\hbar}{2} \left( X (\omega (Y)) + Y (\omega (X)) + 2\omega (X) \cdot Y + 2\omega (Y) \cdot X - \omega \left( \Box_X (Y) + \Box_Y (X) \right) \right) \\ &+ \omega (X) \cdot \omega (Y) + g (X, Y). \end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned} \mathbf{q}_1 (X) \circ \mathbf{q}_1 (Y) &= \mathbf{q}_2 (X \bullet Y) - \frac{\imath\hbar}{2} \mathbf{q}_1 \left( \Box_X (Y) + \Box_Y (X) + [X, Y] \right) \\ &\quad - \frac{\imath\hbar}{2} d\omega (X, Y) - g (X, Y), \end{aligned}$$

and we obtain 3.14 by applying $\mathbf{q}_*^{-1}$.

**Remark 2** *The second formula in (3.14) can be rewritten by using the torsion tensor*

$$\mathbf{T} (X, Y) = \Box_X (Y) - \Box_Y (X) - [X, Y]$$

*in the following way*

$$X \underset{\mathbf{q}}{\bigstar} Y = X \bullet Y - \imath\hbar \Box_X (Y) + \frac{\imath\hbar}{2} \left( \mathbf{T} (X, Y) - d\omega (X, Y) \right) - g (X, Y) \tag{3.17}$$

**Example 4** *Let $g = 0, g_3 = 0$, then*

1. $X \underset{\mathbf{q}}{\bigstar} X = X^2 - \imath\hbar \Box_X (X)$,

   *where $X^2 = X \bullet X$ is the square in the symmetric algebra.*

*2.* $X \underset{\mathbf{q}}{\bigstar} X \underset{\mathbf{q}}{\bigstar} X =$

$X^3 - \imath\hbar X\square_X(X) - \hbar^2\square_X^2(X) + \frac{\hbar^2}{2}(\mathbf{T}(X, \square_X(X)) + d\omega(X, \square_X X))$
*where* $\square_X^2(X) = \square_X \circ \square_X(X)$.

*3.* $X \underset{\mathbf{q}}{\bigstar} X^{k-1} = X^k - \imath\hbar\square_X(X^{k-1})$
*for any vector field* $X$.

**Local coordinates.**
The star-product in local coordinates takes the following form

$$\begin{aligned}
x_a \underset{\mathbf{q}}{\bigstar} x_b &= x_a \cdot x_b, \\
x_a \underset{\mathbf{q}}{\bigstar} p_b &= x_a \cdot p_b, \\
p_b \underset{\mathbf{q}}{\bigstar} x_a &= x_a \cdot p_b - \imath\hbar\delta_{ab}, \\
p_a \underset{\mathbf{q}}{\bigstar} p_b &= p_a \cdot p_b - \frac{\imath\hbar}{2}\left(F_{ab} + \sum_c (\Gamma^c_{ab} + \Gamma^c_{ba})\, p_c\right).
\end{aligned}$$

The first three relations are the common but the fourth one included the "electromagnetic" and the "gravity" terms.

## 3.2 Classical Mechanics

In this section we consider symplectic structures (or classical mechanics) on $T^*M$ that corresponds to described above star-products.

Denote by $\rho \in \Omega^1(T^*M)$ the universal differential 1-form on the cotangent bundle, and let $\Omega = d\rho$ be the standard symplectic structure. Then formula (3.14) shows that the new symplectic structure that corresponds to this deformation is

$$\Omega_c = d\rho + d\omega. \tag{3.18}$$

This structure gives us new mechanical motions (that are motions of charged particles in the case of the Minkowski space).

Namely, for any smooth function $H \in C^\infty(T^*M)$ we can define two vector fields:

1. The usual Hamiltonian field $X_H$ where

$$X_H \rfloor \Omega = dH,$$

and

2. The *charged* Hamiltonian field $X_H^c$ where

$$X_H^c \rfloor \Omega_c = dH.$$

The difference

$$L_H = X_H^c - X_H$$

we will call (generalized) *Lorentz force.*

This is a straightforward generalization of the Lorentz force for charged particles in the Minkowski space for the case of an arbitrary mechanical system.

**Local coordinates.**

Let $(x_1, ..., x_n; p_1, ..., p_n)$ are the natural local coordinate in $T^*M$, and

$$d\omega = \sum_{a,b} F_{a,b} dx_a \wedge dx_b.$$

Then one has

$$X_H = \sum_a \frac{\partial H}{\partial p_a} \frac{\partial}{\partial x_a} - \sum_a \frac{\partial H}{\partial x_a} \frac{\partial}{\partial p_a},$$

and

$$X_H^c = \sum_a \frac{\partial H}{\partial p_a} \frac{\partial}{\partial x_a} - \sum_a \left( \frac{\partial H}{\partial x_a} + \sum_b F_{ab} \frac{\partial H}{\partial p_b} \right) \frac{\partial}{\partial p_a}.$$

Hence, the Lorentz force is

$$L_H = -\sum_{a,b} F_{ab} \frac{\partial H}{\partial p_b} \frac{\partial}{\partial p_a}.$$

For example, if

$$H = \frac{p^2}{2} + U(x)$$

corresponds to the usual mechanical system and the Newton Law

$$\ddot{x}_a = -\frac{\partial U}{\partial x_a}$$

then for "charged" system we have

$$\ddot{x}_a = -\frac{\partial U}{\partial x_a} - \sum_b F_{ab}\,\dot{x}_b\ .$$

## 3.3 Example: Mechanics of $n$-identical particles.

### 3.3.1 General remarks.

In this section we consider the mechanical system of $n-$identical particles on the line. Configuration space $M$ of this system is the factor

$$M = (\mathbb{R}^n \setminus \Sigma)/S_n$$

where $\Sigma = \{(x_1, ..., x_n) \in \mathbb{R}^n |\, \exists a \neq b; x_a = x_b\}$ and $S_n$ is the symmetric group of all permutations of $n$ elements.

Let $x_1, ..., x_n$ be coordinates of the particles. We consider functions

$$\begin{aligned}
&u_1 = x_1 + \cdots + x_n,\\
&u_2 = \frac{1}{2}\left(x_1^2 + \cdots x_n^2\right),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&u_n = \frac{1}{n}\left(x_1^n + \cdots x_n^n\right)
\end{aligned}$$

as coordinates on $M$.

To realize our quantum program we need two connections: connection $\omega$ in the trivial bundle, and connection $\Box$ into the tangent bundle $TM \longrightarrow M$. We choose the Levi-Civita connection $\Box$ associated this the factor-metric (or kinetic energy) of the mechanical system.

Let $(\cdot,\cdot)$ be the usual scalar product. Denote by $\overrightarrow{dx}$ vector with components $dx_1, ..., dx_n$, and by $\overrightarrow{du}$ vector with components $du_1, ..., du_n$.

Then one has

$$\overrightarrow{du} = W(x)\,\overrightarrow{dx}$$

where

$$W(x) = \left\Vert \begin{array}{ccc} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \cdots & \cdots & \cdots \\ x_1^{n-1} & \cdots & x_n^{n-1} \end{array} \right\Vert .$$

Therefore, for the metric $g$ we obtain

$$g = \left(\overrightarrow{dx}, \overrightarrow{dx}\right) = \left(W^{-1}(x)\overrightarrow{du}, W^{-1}(x)\overrightarrow{du}\right) = \left(G^{-1}(u)\overrightarrow{du}, \overrightarrow{du}\right)$$

where $G(u) = W(x)W'(x)$ has the form

$$G(u) = \left\| \begin{array}{ccccc} n & u_1 & 2u_2 & \cdots & (n-1)u_{n-1} \\ u_1 & 2u_2 & 3u_3 & \cdots & nu_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (n-1)u_{n-1} & nu_n & (n+1)u_{n+1} & \cdots & (2n-2)u_{2n-2} \end{array} \right\|$$

and we put

$$u_k = \frac{1}{k}\left(x_1^k + \cdots x_n^k\right)$$

for all $k = 1, 2, ...., 2n-2$.

The Hamiltonian of the system has the form

$$H(u,p) = \frac{1}{2}\left(G(u)p, p\right).$$

### 3.3.2 Two identical particles.

In this case

$$G(u) = \left\| \begin{array}{cc} 2 & u_1 \\ u_1 & 2u_2 \end{array} \right\|$$

and

$$G^{-1}(u) = \frac{1}{\Delta(u)} \left\| \begin{array}{cc} 2u_2 & -u_1 \\ -u_1 & 2 \end{array} \right\|$$

where

$$\Delta(u) = 4u_2 - u_1^2$$

is the discriminant.

Therefore,

$$g = \frac{1}{\Delta(u)}\left(2u_2(du_1)^2 - 2u_1du_1 \bullet du_2 + 2(du_2)^2\right)$$

is the metric on $M$.

We have the following Christoffel symbols

$$\begin{aligned}\Gamma_{11}^{1} &= \Gamma_{12}^{1}=0,\ \Gamma_{22}^{1}=\frac{8u_1}{\Delta(u)};\\ \Gamma_{11}^{2} &= \frac{2u_1^2u_2}{\Delta(u)},\ \Gamma_{12}^{2}=\frac{u_1}{\Delta(u)},\ \Gamma_{22}^{2}=-\frac{2}{\Delta(u)},\end{aligned}$$

and the following star-product

$$\begin{aligned}u_a \underset{\mathbf{q}}{\bigstar} u_b &= u_a\cdot u_b,\ u_a \underset{\mathbf{q}}{\bigstar} p_b = u_a\cdot p_b,\\ p_a \underset{\mathbf{q}}{\bigstar} u_b &= p_a\cdot u_b - \imath\hbar\delta_{ab},\\ p_1 \underset{\mathbf{q}}{\bigstar} p_1 &= p_1^2-\imath\hbar\frac{u_1^2u_2}{\Delta(u)}\cdot p_2,\\ p_1 \underset{\mathbf{q}}{\bigstar} p_2 &= p_2 \underset{\mathbf{q}}{\bigstar} p_1 = p_1\cdot p_2-\imath\hbar\frac{u_1}{\Delta(u)}\cdot p_2,\\ p_2 \underset{\mathbf{q}}{\bigstar} p_2 &= p_2^2-\imath\hbar\left(\frac{8u_1}{\Delta(u)}\cdot p_1-\frac{2}{\Delta(u)}\cdot p_2\right).\end{aligned}$$

We have the following expression for the Hamiltonian

$$H(u,p)=p_1^2+u_1p_1p_2+u_2p_2^2.$$

Therefore, (cf. [5])

$$\begin{aligned}\widehat{H} &= \mathbf{q}_2(H)=-\hbar^2\left(\frac{\partial^2}{\partial^2 u_1}+u_1\frac{\partial^2}{\partial u_1\partial u_2}+u_2\frac{\partial^2}{\partial^2 u_2}\right)+\\ &\quad \frac{8\hbar^2u_1u_2}{\Delta}\frac{\partial}{\partial u_1}+\frac{\hbar^2\left(u_1^2+2u_1^2u_2-2u_2\right)}{\Delta}\frac{\partial}{\partial u_2}.\end{aligned}$$

### 3.3.3 Three identical particles.

In this case

$$G(u)=\left\|\begin{array}{ccc}3 & u_1 & 2u_2\\ u_1 & 2u_2 & 3u_3\\ 2u_2 & 3u_3 & 4u_4\end{array}\right\|$$

where

$$u_4=\frac{1}{24}u_1^4+u_1u_3-\frac{1}{2}u_1^2u_2+\frac{1}{2}u_2^2.$$

Therefore,

$$G^{-1}(u) = \frac{1}{\Delta(u)} \left\| \begin{array}{ccc} 8u_2u_4 - 9u_3^2 & 6u_2u_3 - 4u_1u_4 & 3u_1u_3 - 4u_2^2 \\ 6u_2u_3 - 4u_1u_4 & 12u_4 - 4u_2^2 & 2u_1u_2 - 9u_3 \\ 3u_1u_3 - 4u_2^2 & 2u_1u_2 - 9u_3 & 6u_2 - u_1^2 \end{array} \right\|$$

where

$$\Delta(u) = 24u_2u_4 - 27u_3^2 - 4u_1^2u_4 + 12u_1u_2u_3 - 8u_2^3$$

is the discriminant.

The metric $g$ has the following form

$$\Delta(u) \cdot g = (8u_2u_4 - 9u_3^2)(du_1)^2 + 2(6u_2u_3 - 4u_1u_4)\, du_1 \bullet du_2 + \\ 2(3u_1u_3 - 4u_2^2)\, du_1 \bullet du_3 + (12u_4 - 4u_2^2)(du_2)^2 + 2(2u_1u_2 - 9u_3)\, du_2 \bullet du_3 \\ + (6u_2 - u_1^2)(du_3)^2$$

## 3.4 The sphere.

In this section we consider the problem of quantization of mechanical systems on 2-dimensional sphere $S^2$. We choose the trivial connection $\omega = 0$ and the Levi-Civita connection associated with the usual metric

$$g = \cos^2\psi \ (d\varphi)^2 + (d\psi)^2$$

where $\varphi$ and $\psi$ are the spherical coordinates, $0 < \varphi < 2\pi$, $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$.

In this case we have the following Christoffel symbols

$$\Gamma_{12}^1 = -\tan\psi,\ \Gamma_{11}^2 = \frac{1}{2}\sin(2\psi), \quad \text{and} \quad \Gamma_{ab}^c = 0 \text{ in the other cases.}$$

Let $p_\varphi$ and $p_\psi$ momenta corresponding to $\varphi$ and $\psi$.

One has the following star-multiplications

$$\begin{array}{l} \varphi \underset{\mathbf{q}}{\bigstar} p_\varphi = \varphi \cdot p_\varphi,\ \ \psi \underset{\mathbf{q}}{\bigstar} p_\varphi = \psi \cdot p_\varphi, \\ \varphi \underset{\mathbf{q}}{\bigstar} p_\psi = \varphi \cdot p_\psi,\ \ \psi \underset{\mathbf{q}}{\bigstar} p_\psi = \psi \cdot p_\psi, \\ p_\varphi \underset{\mathbf{q}}{\bigstar} \varphi = \varphi \cdot p_\varphi - \imath\hbar,\ \ p_\varphi \underset{\mathbf{q}}{\bigstar} \psi = \psi \cdot p_\varphi, \\ p_\psi \underset{\mathbf{q}}{\bigstar} \psi = \psi \cdot p_\psi - \imath\hbar,\ \ p_\psi \underset{\mathbf{q}}{\bigstar} \varphi = \varphi \cdot p_\psi, \end{array}$$

$$p_\psi \underset{\mathbf{q}}{\bigstar} p_\varphi = p_\varphi \underset{\mathbf{q}}{\bigstar} p_\psi = p_\varphi \cdot p_\psi + \imath\hbar \tan\psi \cdot p_\varphi,$$
$$p_\varphi \underset{\mathbf{q}}{\bigstar} p_\varphi = p_\varphi^2 - \frac{\imath\hbar}{2}\sin(2\psi) \cdot p_\psi,$$
$$p_\psi \underset{\mathbf{q}}{\bigstar} p_\psi = p_\psi^2.$$

The Hamiltonian of the mechanical system has the form

$$H = \frac{1}{2}\left(\frac{p_\varphi^2}{\cos^2(\psi)} + p_\psi^2\right).$$

Therefore,

$$\widehat{H} = \mathbf{q}_2(H) = -\frac{\hbar^2}{2}\left(\frac{1}{\cos^2(\psi)}\frac{\partial^2}{\partial^2\varphi} + \frac{\partial^2}{\partial^2\psi}\right) + \frac{\hbar^2}{4}\tan(\psi)\frac{\partial}{\partial\psi}.$$

## 3.5 The Lobachevsky plane.

Consider the Poincare model of the Lobachevsky plane. Here $M$ is the upper half-plane

$$\mathbb{R}_+^2 = \left\{(x_1, x_2) \in \mathbb{R}^2 \middle| \, x_2 > 0\right\}$$

equipped with the metric

$$g = \frac{(dx_1)^2 + (dx_2)^2}{x_2^2}.$$

In this case we have the only non-trivial Christoffel symbols

$$\Gamma_{12}^1 = \Gamma_{22}^2 = -x_2^{-1}, \quad \Gamma_{11}^2 = x_2^{-1}.$$

Therefore the star-product has the form

$$\begin{aligned}
x_a \underset{\mathbf{q}}{\bigstar} x_b &= x_a \cdot x_b, \; x_a \underset{\mathbf{q}}{\bigstar} p_b = x_a \cdot p_b, \\
p_a \underset{\mathbf{q}}{\bigstar} x_b &= p_a \cdot x_b - \imath\hbar\delta_{ab}, \\
p_1 \underset{\mathbf{q}}{\bigstar} p_1 &= p_1^2 - \frac{\imath\hbar}{x_2} p_2, \\
p_1 \underset{\mathbf{q}}{\bigstar} p_2 &= p_2 \underset{\mathbf{q}}{\bigstar} p_1 = p_1 \cdot p_2 + \frac{\imath\hbar}{x_2} p_1, \\
p_2 \underset{\mathbf{q}}{\bigstar} p_2 &= p_2^2 + \frac{\imath\hbar}{x_2} p_2.
\end{aligned}$$

We have the following expressions for the Hamiltonian

$$H = \frac{x_2^2}{2}\left(p_1^2 + p_2^2\right)$$

and the quantum image

$$\widehat{H} = -\frac{\hbar^2 x_2^2}{2}\left(\partial_1^2 + \partial_2^2\right).$$

# Chapter 4

# Quantizations of Mechanical Systems equipped with inner structures

## 4.1 Quantizations

Let $\alpha : E(\alpha) \longrightarrow M$ and $\beta : E(\beta) \longrightarrow M$ be smooth vector bundles (over $\mathbb{C}$). Denote by $C^{\infty}(\alpha)$ the correspondent $\mathcal{F}$-module of smooth sections of $\alpha$.

Let $\mathbf{Diff}_k(\alpha,\beta)$ be a $\mathcal{F}$-module of differential operators of order $\leq k$ acting from $C^{\infty}(\alpha)$ to $C^{\infty}(\beta)$, and

$$\mathbf{Diff}_*(\alpha,\beta) = \bigcup_{k\geq 0} \mathbf{Diff}_k(\alpha,\beta)$$

be a module of all differential operators.

In the given case the Hamiltonian map is

$$\mathbf{Ham}_k\colon \mathbf{Diff}_k(\alpha,\beta) \longrightarrow \mathbf{Hom}(\alpha,\beta)\otimes \mathbf{S}^k(\mathbf{D})$$

where $\mathbf{Hom}(\alpha,\beta)$ is a $\mathcal{F}$-module of $\mathcal{F}$-morphisms from $\alpha$ to $\beta$.

Moreover, as above, we have the following exact sequences

$$\mathbf{0} \longrightarrow \mathbf{Diff}_{k-1}(\alpha,\beta) \longrightarrow \mathbf{Diff}_k(\alpha,\beta) \underset{\mathbf{Ham}_k}{\longrightarrow} \mathbf{Hom}(\alpha,\beta)\otimes \mathbf{S}^k(\mathbf{D}) \longrightarrow \mathbf{0}.$$

**Remark 3** ***Local coordinates.***

*Let $x_1, ..., x_n$ be a local coordinates in a neighbourhood $\mathcal{O} \subset M$; $e_1^\alpha, ..., e_m^\alpha$; $e_1^\beta, ..., e_{m'}^\beta$ be bases in sections of $\alpha$ and $\beta$ over $\mathcal{O}$.*

*Any operator $\Delta \in \mathbf{Diff}_k(\alpha, \beta)$ over $\mathcal{O}$ might be represented in the matrix form*

$$\Delta = \sum_{|\sigma| \leq k} A_\sigma(x)\, \partial^\sigma$$

*where $A_\sigma(x) \in \mathbf{Hom}(\alpha|_{\mathcal{O}}, \beta|_{\mathcal{O}})$ are smooth matrix valued functions.*

*The Hamiltonian map has a form*

$$\mathbf{Ham}_k{:}\Delta \longmapsto H_\Delta = \sum_{|\sigma|=k} A_\sigma(x) \left(\frac{\imath}{\hbar} p\right)^\sigma .$$

We define a *quantization* as a $\mathcal{F}-$linear morphisms

$$\mathbf{q}_k : \mathbf{Hom}(\alpha, \beta) \otimes \mathbf{S}^k(\mathbf{D}) \longrightarrow \mathbf{Diff}_k(\alpha, \beta)$$

such that

1. $\mathbf{q}_k$ determined for all smooth vector bundles $\alpha$'s and $\beta$'s.
2. $\mathbf{q}_0 = \mathrm{id}$,
3. $\mathbf{q}_* = \sum_{k \geq 0} \mathbf{q}_k$ are $\mathbf{Hom}(*, *)-$morphisms,

   that is $\mathbf{q}_*(A \circ \theta) = A \circ \mathbf{q}_*(\theta)$ for all $\theta \in \mathbf{Hom}(\alpha, \beta) \otimes \mathbf{S}^k(\mathbf{D})$, and $A \in \mathbf{Hom}(\beta, \gamma)$,
4. $\mathbf{Ham}_k \circ \mathbf{q}_k = \mathrm{id}$.

Consider quantizations on the level $k = 1$.

As above one gets the following

**Theorem 2** *Any quantization $\mathbf{q}_1 : \mathbf{Hom}(\alpha, \beta) \otimes \mathbf{D} \longrightarrow \mathbf{Diff}_1(\alpha, \beta)$ has the form*

$$\mathbf{q}_1 : A \otimes X \longmapsto \frac{\hbar}{\imath} A \circ \nabla_X^\alpha$$

*where $X \in \mathbf{D}$, $A \in \mathbf{Hom}(\alpha, \beta)$, and $\nabla_X^\alpha \in \mathbf{Diff}_1(\alpha, \alpha)$ is a covariant derivative along vector field $X$ with respect to some linear connection in the vector bundle $\alpha$.*

To get the Heisenberg relations one should compare two expressions

$$\mathbf{q}_1([X,Y]) = \frac{\hbar}{\imath}\nabla^{\alpha}_{[X,Y]}$$

and

$$[\mathbf{q}_1(X), \mathbf{q}_1(Y)] = -\hbar^2 [\nabla^{\alpha}_X, \nabla^{\alpha}_Y].$$

This gives immediately the following

**Theorem 3** *In the Lie algebra* $\mathbf{Diff}_1(\alpha,\alpha)$ *one has the following Heizenberg relations*

$$\frac{\imath}{\hbar}[\mathbf{q}_1(X), \mathbf{q}_1(Y)] - \mathbf{q}_1([X,Y]) = \mathcal{R}_{\nabla}(X,Y) \tag{4.1}$$

*where* $X, Y \in \mathbf{D}$, *and* $\mathcal{R}_{\nabla} \in \Omega^2(M) \otimes \mathbf{End}(\alpha)$ *is the curvature of the connection* $\nabla^{\alpha}$.

For the level $k = 2$ we have the following description.

**Theorem 4** *1. Any quantization* $\mathbf{q}_2 : \mathbf{Hom}(\alpha,\beta) \otimes \mathbf{S}^2(\mathbf{D}) \longrightarrow \mathbf{Diff}_2(\alpha,\beta)$ *has the form*

$$\begin{aligned}\mathbf{q}_2 : A \otimes X \bullet Y \longmapsto & \frac{1}{2} A \circ (\mathbf{q}_1(X) \circ \mathbf{q}_1(Y) + \mathbf{q}_1(Y) \circ \mathbf{q}_1(X)) \\ & + \frac{\imath\hbar}{2} A \circ \mathbf{q}_1(\square_X Y + \square_Y X) + A \circ g(X,Y)\end{aligned}$$

*where* $\square$ *is a linear connection in the tangent bundle, and we identified vector fields* $Z \in \mathbf{D}$ *with* $\mathrm{id} \otimes Z \in \mathbf{Hom}(\alpha,\alpha) \otimes \mathbf{D}$. *So* $\mathbf{q}_1(X)$ *means* $\mathbf{q}_1(\mathrm{id} \otimes X) \in \mathbf{Diff}_1(\alpha,\alpha)$, *and* $g : \mathbf{S}^2(\mathbf{D}) \longrightarrow \mathbf{End}(\alpha)$ *is an arbitrary symmetric 2-tensor with values into* $\mathbf{End}(\alpha)$.

*2. Quantizations* $\mathbf{Hom}(\alpha,\beta) \otimes \mathbf{S}^k(\mathbf{D}) \longrightarrow \mathbf{Diff}_k(\alpha,\beta)$ *on the decomposable elements has the form*

$$\begin{aligned}\mathbf{q}_k(A \otimes X_1 \bullet \cdots \bullet X_k) = & \frac{1}{k} A \circ \sum_{a=1}^{k} \mathbf{q}_1(X_a) \circ \mathbf{q}_{k-1}\left(X_{(a)}\right) + \\ & \frac{\imath\hbar}{k} A \circ \mathbf{q}_{k-1}\left(\sum_{a=1}^{k} \sum_{b \neq a} \square_{X_a}(X_b)\left(X_{(a)}\right)_{(b)}\right) + A \circ g_{k-2}(X_1, ..., X_k\end{aligned} \tag{4.2}$$

*for some morphisms* $g_{k-2} : \mathbf{S}^k(\mathbf{D}) \longrightarrow \mathbf{Diff}_{k-2}(\alpha,\alpha)$.

**Remark 4** ***Local coordinates.*** *($g_{k-2}=0,\ k=2,3...$).*

*In local coordinates one has*

$$\nabla^{\alpha}_{\partial_a}\left(e^{\alpha}_{b}\right)=\sum_{c}\vartheta^{c}_{ab}e^{\alpha}_{c}$$

*for some matrices* $\boldsymbol{\vartheta}_a=\|\vartheta^{c}_{ab}\|\in\mathbf{End}\left(\alpha\right).$

*Then*

$$\mathbf{q}_1 : A\left(x\right)p_s\longmapsto\frac{\hbar}{\imath}A\left(x\right)\left(\partial_s+\boldsymbol{\vartheta}_s\right)$$

*for all matrix valued functions* $A\left(x\right)$*.*

*Self-adjointness of the operators* $\frac{\hbar}{\imath}\left(\partial_s+\boldsymbol{\vartheta}_s\right)$ *gives the skew-symmetry conditions on* $\boldsymbol{\vartheta}_s$*, that is*

$$\boldsymbol{\vartheta}^{*}_{s}=-\boldsymbol{\vartheta}_s.$$

*In other words, this means that all vector bundles should be hermitian bundles and* $\nabla^{\alpha}$ *should be hermitian connections.*

*The Heisenberg relations takes now form (4.1) where*

$$\mathcal{R}_{\nabla}\left(X,Y\right)=d\boldsymbol{\vartheta}\left(X,Y\right)+\left[\boldsymbol{\vartheta}\left(X\right),\boldsymbol{\vartheta}\left(Y\right)\right]\tag{4.3}$$

*and*

$$\boldsymbol{\vartheta}=\sum_{s}dx_s\otimes\boldsymbol{\vartheta}_s\in\Omega^{1}\left(M\right)\otimes\mathbf{End}\left(\alpha\right).\tag{4.4}$$

*For quantizations on the level* $k=2$ *one has*

$$\begin{aligned}\mathbf{q}_2 : A\left(x\right)p_sp_r\longmapsto&-\frac{\hbar^2}{2}A\circ\left(\left(\partial_s+\boldsymbol{\vartheta}_s\right)\circ\left(\partial_r+\boldsymbol{\vartheta}_r\right)+\left(\partial_r+\boldsymbol{\vartheta}_r\right)\circ\left(\partial_s+\boldsymbol{\vartheta}_s\right)\right)\\&+\frac{\hbar^2}{2}A\circ\sum_{t}\left(\Gamma^{t}_{sr}+\Gamma^{t}_{rs}\right)\left(\partial_t+\boldsymbol{\vartheta}_t\right)\end{aligned}$$

## 4.2 Example. Mechanics of $n-$identical particles. Continuation.

In this section we consider the quantization of the above problem as an example of mechanical system equipped with inner structure.

We give a geometrical reformulation of this problem. The physical meaning of this approach can be found in ([2],[6],[4]).

Let $M_0$ be a configuration space for some mechanical system or particle. The configuration space of $n-$identical such systems can be described as the quotient

$$M = ((M_0)^n \setminus \Sigma_n) / S_n$$

where

$$(M_0)^n = \underbrace{M_0 \times \cdots \times M_0}_{n-\text{times}}$$

and

$$\Sigma_n = \{(x_1, ..., x_n) \in (M_0)^n \,|\, \exists a \neq b;\ x_a \neq x_b\}.$$

Sometimes (specially in fractional statistics) we will need some covering of $((M_0)^n \setminus \Sigma_n)$, say $X_n \longrightarrow ((M_0)^n \setminus \Sigma_n)$. Such coverings gives a possibility to consider ramifications of wavefunctions as functions on $X_n$. We will unify the situation into a covering

$$\varkappa : X_n \longrightarrow ((M_0)^n \setminus \Sigma_n) \longrightarrow M \tag{4.5}$$

with a structure group $G$ equipped with a representation $\xi : G \longrightarrow S_n$ corresponding to the first arrow in (4.5).

Denote by $F(x) = \text{Functions}(\varkappa^{-1}(x))$ the totality of all $\mathbb{C}$-valued functions on the fibre $\varkappa^{-1}(x)$. This algebra might be considered as algebra of all values of possible (ramificated) wavefuncions at the point $x \in M$.

Let $\mathcal{S}$ be an irreducible $G-$module. We say that on the manifold $M$ defined a *statistics of type* $\mathcal{S}$ if on $M$ given a vector bundle $\pi_{\mathcal{S}} : E \longrightarrow M$ such that

1. $E(x) = \pi_{\mathcal{S}}^{-1}(x) \subset F(x)$, and
2. $E(x)$ is an irreducible $G-$module of the type $\mathcal{S}$,

for all $x \in M$.

We will identify sections of the bundle $\pi_{\mathcal{S}}$ with wavefunctions of correspondent quantum problem that have a ramification of the type $G$ and satisfying statistics $\mathcal{S}$.

To realize our quantum paradigm for this case we need two connections.

We can choose the Levi-Civita connection $\square$ in the tangent bundle $TM \longrightarrow M$ that corresponds to the factor metric. In this case quantizations will correspond to linear connections in the vector bundle $\pi_{\mathcal{S}}$.

**Example 5** *Let* $X_n = ((M_0)^n \setminus \Sigma_n)$, *and* $G = S_n$. *This means that we consider wavefunctions without ramification. There are two* $1-$*dimensional representations of* $S_n$ : *(1)the trivial one ("bosons"), and (2) the sign representation ("fermions"). Multidimensional representations correspond to non trivial unitary solutions of the Yang-Baxter equation.*

**Example 6** *Let* $M_0 = \mathbb{R}^2$ *and* $X_n \longrightarrow ((M_0)^n \setminus \Sigma_n)$ *be the universal covering. Then* $G$ *corresponds to the braid group, and the above procedure describes statistics in terms of irreducible representations of the braid group, or in terms of non-unitary solutions of the Yang-Baxter equation.*

**Remark 5** *In paper ([4]) described the algebra of* $S_N-$*invariants into* $S_N$-*extended Heisenberg algebra. In our terminology this algebra coinsides with quantization of algebra* $\mathbf{End}(\pi_{\mathcal{S}}) \otimes \mathbf{S}^*(\mathbf{D})$ *or, in other words, with algebra of all differential operators* $\mathbf{Diff}_*(\pi_{\mathcal{S}}, \pi_{\mathcal{S}})$. *Therefore, the problem of description all representations of the algebra (or, in other terms, fractional statistics) is a problem of* $\pi_{\mathcal{S}}-$*differential equations.*

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