Nijenhuis tensors and obstructions for pseudoholomorphic mapping constructions

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Abstract

In this paper we present some approaches to classification of almost complex structures and to construction of local or formal pseudoholomorphic mapping from one almost complex manifold to another. The corresponding criteria are given in terms of Nijenhuis tensors and their generalizations.

We deal with the prolongations of the Cauchy-Riemann equation in 1-jets of the mapping from one almost complex manifold to another. We give a criterion of the prolongation existence of k-th pseudoholomorphic jets to the $(k+1)$-th ones. As a consequence basing on the Kuranishi's approach we obtain formal normal forms of almost complex structures. As another consequence we get the formal part of Newlander-Nirenberg's theorem on integrability.

We also introduce the notion and study the properties of the linear Nijenhuis tensor, which we apply to almost complex structures of general position. Two applications are exhibited: a classification of four-dimensional almost complex structures in terms of distributions and a characterization of the new class of structures which we call Lie almost complex structures.

Contents

Introduction

Let (L^{2l}, j_L) and (M^{2m}, j_M) be almost complex manifolds, i.e. manifolds equipped with automorphisms j_L and j_M of the tangent bundles which satisfy the conditions $j_L^2 = -\mathbf{1}_L : T_*L \to T_*L$ and $j_M^2 = -\mathbf{1}_M$. A pseudoholomorphic (PH-)mapping $u : L \to M$ is such a mapping that its differential $u_*|_x = du_x : T_xL \to T_{u(x)}M$ commutes with the almost complex structure: $j_M \circ u_* = u_* \circ j_L$.

If $M = \mathbb{C}$ a (nontrivial) pseudoholomorphic mapping $u : L \to M$ is called a complex coordinate and in general case there is no such a coordinate (moreover if on a manifold L^{2l} there exist l independent local complex coordinates then it is complex). For $l = 1$, i.e. when (L, j_L) is a Riemann surface, PH-mappings do exist and moreover the family of such mappings possesses the manifold structure (under some additional conditions, see [Gr1], [MS]). If $l > 1$ then for an almost complex structure j_L of general position such a mapping does not exist even locally. An obstruction is the Nijenhuis tensor. It is well-known ([NN], [NW]) that if the Nijenhuis tensor equals zero identically then the almost complex manifold is complex and for a pair of such manifolds L and M there exists a local holomorphic mapping $u : (L, x) \to (M, y)$ (for more see [K]). In this paper we consider the existence conditions of formal and local pseudoholomorphic mappings for general almost complex structures.

It is worth mentioning that the existence of a (nontrivial) pseudoholomorphic mapping is a rare situation for mappings $u : L \to M$ between almost complex manifolds (L^{2l}, j_L) and (M^{2m}, j_M) in general position unless $l = 1$. For example, for any $N \geq 1$ the pseudoholomorphic imbedding to the standard complex space $(L^{2l}, j_L) \to (\mathbb{C}^N, j_0)$ exists if and only if the structure j_L is integrable, i.e. complex, which is of course not the case of general position. This rigidity result exhibits a great difference between almost complex structures and some other tensor fields such as Riemannian metrics. Recall that according to the Nash's theorem [N] any Riemannian manifold admits an isometric imbedding $(V, g) \to (\mathbb{R}^N, g_0)$ to the Euclidean space of some big dimension $N \gg 1$ (see also [Gr2]). Thus any pseudoholomorphic mapping $(L^{2l}, j_L) \to (M^{2m}, j_M)$, $l > 1$, is very unstable under the perturbation of almost complex structures, and hence due to the strict rigidity of local pseudoholomorphic mappings the problem generally goes onto the formal level.

The structure of the paper is the following. In chapter 1 we prove a necessary and sufficient condition for the existence of a prolongation of a pseudoholomorphic on 1-jet level mapping of one almost complex manifold to another to a pseudoholomorphic mapping on 2-jet level. In other words we describe the first prolongation of the corresponding Cauchy-Riemann equation on the 1st-jet level. In this case the Cauchy-Riemann equation is the commutation condition of the required mapping u differential $\Phi = du$ at a point with the almost complex structures: $j_M \circ \Phi = \Phi \circ j_L$. We introduce the space of k-jets of pseudoholomorphic mappings which does not have the manifold structure in general case but does have for formally integrable "completed" Cauchy-Riemann equations. Here by "completion" we mean an addition of defining relations, i.e. a contraction from the point of view of jet spaces ([KLV]). The Cauchy-Riemann equation for mappings of one almost complex manifold to another is formally integrable in the integrable case only, i.e. when the manifolds are complex. In general case this equation must be "completed". The first term of "completion" is the commutation condition of the required mapping differential with the Nijenhuis tensors: $N_{j_M} \circ \wedge^2 \Phi = \Phi \circ N_{j_L}$. In chapter 2 we consider the question of formal integrability, introduce a new invariant — Nijenhuis tensor of the second order $N_j^{(2)}$ and write down the next term of "completion": $N_{j_M}^{(2)} \circ \wedge^2(\wedge^2 \Phi) = \Phi \circ N_{j_L}^{(2)}$.

In chapter 3 we prove the solvability criterion for the Cauchy-Riemann equation on the higher jet spaces. As a corollary we get the formal part of the well-known Newlander-Nirenberg theorem on the integrability of almost complex structures. We also use this criterion to find formal normal forms of almost complex structures.

In chapter 4 we introduce the space of linear Nijenhuis tensors and consider the recovery question for liner complex structure by a Nijenhuis tensor. In chapter 5 we define the notion of the Nijenhuis tensors of general position. We call an almost complex structure the structure of general positions if the corresponding Nijenhuis tensor is in general position. For the manifold equipped with such structures the necessary and sufficient condition for existence of a pseudoholomorphic mapping may be formulated in terms of Nijenhuis tensors only, which forms the statement of theorem 6.

In section 6.1 we present a classification of almost complex structures of general positions on four-dimensional manifolds in terms of distributions. In section 6.2 we introduce the notion of Lie almost complex structure and present a classification of such structures of general position.

In appendix we prove that the notion general position for the Nijenhuis tensors, introduced in chapter 3, actually satisfies the general position properties.

Chapters 4–6 do not use the results and methods of chapters 1–3 and might be read independently.

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Chapter 1

Nijenhuis tensor as the first obstruction to the construction of pseudoholomorphic mapping

For the mapping $u : (L, x) \to (M, y)$ we denote by $u_*^{\wedge k} : \wedge^k T_x L \to \wedge^k T_y M$ the induced mapping.

Let us denote by $\mu = \mu_x$ the ideal in the ring of functions $C^\infty(L)$ vanishing at the point x. The power of this ideal μ^k is the ideal in the ring of functions vanishing at the point x together with their derivatives of the order $< k$: $\mu^k = \{f | \partial^\sigma f(x) = 0,\ |\sigma| < k\}$. Let's denote by $\mu^k = \mu^k T(p, q)$ the submodule in the module of tensors of the type (p, q) vanishing at the point x up to the order k. Denote by the symbol $\mathcal{O}_L(x)$ the germ of neighborhoods of the point $x \in L$. Let us also denote $\mu^k_{x,y} = \{\Phi : T\mathcal{O}_L(x) \to T\mathcal{O}_M(y) | \operatorname{Im}\Phi \subset \mu^k_y\}$.

A formal pseudoholomorphic mapping is a mapping $u : \mathcal{O}_L(x) \to \mathcal{O}_M(y)$ such that $j_M \circ u_* - u_* \circ j_L \in \mu^\infty$. The mapping is called nontrivial if $u_* \notin \mu^\infty$. Moreover we will consider, in general, nondegenerate mappings: $u_* \notin \mu$ or $du_x \neq 0$.

Let us consider the Nijenhuis tensors N_{j_L} and N_{j_M} on manifolds L and M:

$$N_j(\xi, \eta) = [j\xi, j\eta] - j[j\xi, \eta] - j[\xi, j\eta] - [\xi, \eta].$$

Immediately from the definition it follows that if $u : L \to M$ is a PH-mapping then $N_{j_M} \circ u_*^{\wedge 2} = u_* \circ N_{j_L}$. Moreover this equality holds at the point x if u preserves the almost complex structure up to the second order: $j_M \circ u_* - u_* \circ j_L \in \mu^2_{x,y}$.

Let us denote $J^0_{PH}(L,M) = J^0(L,M) = L \times M$ (we will use the notations from the jet spaces theory, see [KLV], [KS]). Let us denote by $J^1_{PH}(L,M)$ the subbundle of the jet bundle $J^1(L,M) \to J^0(L,M)$ with the fiber

$$(J^1_{PH})_{x,y} = \{\Phi : T_xL \to T_yM \,|\, j_M \circ \Phi = \Phi \circ j_L\}.$$

In other words the manifolds J^1_{PH} is a bundle over $L \times M$ with the fiber $(J^1_{PH})_{x,y}$ being the factor space of the space $\{u_* : T\mathcal{O}_L(x) \to T\mathcal{O}_M(y) \,|\, j_M \circ u_* - u_* \circ j_L \in \mu_{x,y}\}$ by the space $\{u_* \in \mu_{x,y}\}$. Let us also denote

$$J^k_{PH} \supset (J^k_{PH})_{x,y} = \frac{\{u_* : T\mathcal{O}_L(x) \to T\mathcal{O}_M(y) \,|\, u_* \circ j_L - j_M \circ u_* \in \mu^k_{x,y}\}}{\{u_* : T\mathcal{O}_L(x) \to T\mathcal{O}_M(y) \,|\, u_* \in \mu^k_{x,y}\}}.$$

Unlike J^1_{PH} the set J^k_{PH}, $k \geq 2$, does not have in general the manifold structure. This is connected with the fact that for the mapping $\pi^{r,s}_{PH}$ of the restriction of the projection $\pi^{r,s} : J^r(L,M) \to J^s(L,M)$, $r \geq s$, the inverse image might be empty: $(\pi^{r,s}_{PH})^{-1}(pt) = \emptyset$.

We apply the Cartan's method of prolongations-projections ([ALV]) to obtain a criterion of existence of a formal PH-mapping. Denote by $\mathcal{E}^k$ the image of the projection $\operatorname{Im} \pi^{k,1}_{PH} \subset J^1_{PH} \subset J^1(L,M)$. One may consider these subsets in the 1-jet spaces as differential equations of the first order. The problem of formal pseudoholomorphic mapping construction is reduced to the contraction of the equation — to the finding of a projective limit

$$\mathcal{E}^\infty \subset \ldots \mathcal{E}^2 \subset \mathcal{E}^1 \subset J^1(L,M).$$

The equation $\mathcal{E}^1 = J^1_{PH}$ is fibered over $J^0 = L \times M$ with the fiber

$$\mathcal{E}^1_{x,y} = (\pi^{1,0}_{PH})^{-1}(x,y) = \{\Phi : T_xL \to T_yM \,|\, j_M \circ \Phi = \Phi \circ j_L\}.$$

Theorem 1. *The restriction of the projection $\pi^{1,0}_{PH}$ to the equation $\mathcal{E}^2 \subset J^1_{PH}$ has the inverse images*

$$\mathcal{E}^2_{x,y} = \{\Phi : T_xL \to T_yM \,|\, j_M \circ \Phi = \Phi \circ j_L,\ N_{j_M} \circ \Phi^{\wedge 2} = \Phi \circ N_{j_L}\}.$$

Notations. For the proof of this and further theorems we need the calculations in jet spaces. For this purpose we introduce the following constructions.

Definition 1. Let us define for arbitrary vector field ξ on X the mapping $d^p\xi = d^p_\nabla\xi : \otimes_1^p T_xX \to T_xX$, $x \in X$ by the formula

$$d\xi(\eta) = \nabla_\eta \xi,\ d^2\xi(\eta_1,\eta_2) = \nabla_{\eta_2}\nabla_{\eta_1}(\xi) - \nabla_{\nabla_{\eta_2}\eta_1}(\xi), \quad \text{and for arbitrary } p:$$

$$\begin{aligned} &d^{p+1}\xi(\eta_1,\dots,\eta_{p+1}) \\ &\qquad = d\left(d^p\xi(\eta_1,\dots,\eta_p)\right)(\eta_{p+1}) - \textstyle\sum_{i=1}^p d^p\xi(\eta_1,\dots,d\eta_i(\eta_{p+1}),\dots,\eta_p), \end{aligned}$$

where ∇ is some symmetric connection. If the curvature of the connection ∇ vanishes then the differential $d^p_\nabla\xi$ is symmetric, $d^p_\nabla\xi : S^pT_xX \to T_xX$, and in local Euclidean coordinates it has the form:

$$d^p\xi(\eta_1,\dots,\eta_p)^i = \frac{\partial^p\xi^i}{\partial x^{i_1}\dots\partial x^{i_p}}\eta_1^{i_1}\dots\eta_p^{i_p},$$

and it's easy to see that for $d\xi = d^1\xi$ holds $L_\xi\eta = [\xi,\eta] = d\eta(\xi) - d\xi(\eta)$.

Definition 2. For arbitrary morphism of the bundles $A = (A_x)_{x\in X}$, $A_x : \wedge^qT_xX \to T_xX$, let us define the mapping d^pA, $d^pA_x : \wedge^qT_xX \otimes (\otimes_1^pT_xX) \to T_xX$ by induction: $\quad d^0A := A, \quad d^{p+1}A(\xi_1,\dots,\xi_{q+p+1}) =$

$$= d\left(d^pA(\xi_1,\dots,\xi_{q+p})\right)(\xi_{q+p+1}) - \sum_{i=1}^{q+p} d^pA(\xi_1,\dots,d\xi_i(\xi_{q+p+1}),\dots,\xi_{q+p}).$$

If the curvature of the connection ∇ vanishes (which always might be supposed locally true) then the differential $d^pA_x : \wedge^qT_xX \otimes S^pT_xX \to T_xX$ has the following form in local coordinates:

$$[d^pA(\xi_1,\dots,\xi_{q+p})]^i = \sum_{i_1<\dots<i_q} \frac{\partial^pA^i_{i_1\dots i_q}}{\partial x^{i_{q+1}}\dots\partial x^{i_{q+p}}}\xi_1^{i_1}\dots\xi_q^{i_q}\dots\xi_{q+p}^{i_{q+p}}.$$

Thus a Euclidean connection ∇ defines the mapping

$$d^p_\nabla : \Omega^q \otimes \mathcal{D} \to \Omega^q \otimes S^p\Omega^1 \otimes \mathcal{D},$$

where $\mathcal{D}$ is the module of vector fields, and Ω^q is the module of exterior forms. Let us note that such a mapping might be constructed by any connection ∇ ([P]), but as far as all the arguments below using ∇ are local we will suppose in general that this connection is trivial (Euclidean). We will write d^p instead of d^p_∇.

Proof of theorem 1. 1-PH-jet Φ is lifted to 2-PH-jet iff the following equations hold:

$$\begin{cases} j_M \circ \Phi(\xi) = \Phi \circ j_L(\xi), & \Phi^{(2)}(\xi,\eta) = \Phi^{(2)}(\eta,\xi), \\ dj_M(\Phi\xi,\Phi\eta) + j_M \circ \Phi^{(2)}(\xi,\eta) = \Phi^{(2)}(j_L\xi,\eta) + \Phi \circ dj_L(\xi,\eta), \end{cases} \tag{1}$$

where $\xi,\eta \in T_xL$ and $\Phi^{(2)} : S^2T_xL \to T_yM$ is the 2-symbol of the required PH-mapping (in local coordinates $\Phi^{(2)}(\xi,\eta)^i = \dfrac{\partial^2 u^i}{\partial x^r \partial x^s}\xi^r\eta^s$). Let us consider the tensor

$$P \in T^*_xL \otimes T^*_xL \otimes T_yM,\ \ P(\xi,\eta) = j_M \circ \Phi^{(2)}(\xi,\eta) - \Phi^{(2)}(j_L\xi,\eta). \tag{2}$$

From (1) we have:

$$P(\xi,\eta) = \Phi \circ dj_L(\xi,\eta) - dj_M(\Phi\xi,\Phi\eta). \tag{3}$$

Lemma 1. *The tensor P defined by a symmetric tensor $\Phi^{(2)}$ by the formula (2) satisfies the equations*

$$(i)\qquad j_M \circ P(\xi,\eta) = -P(j_L\xi,\eta), \tag{4}$$

$$(ii)\qquad P(\xi,\eta) - P(\eta,\xi) = P(j_L\xi,j_L\eta) - P(j_L\eta,j_L\xi). \tag{5}$$

Lemma 2. *Every tensor satisfying the system (4)-(5) may be represented in the form (2) where $\Phi^{(2)}$ is a symmetric tensor.*

Proof of lemma 2. Since any bilinear map is decomposed to the sum of (j_L-j_M-)linear and antilinear by the first argument maps it follows from (4) that

$$P(\xi,\eta) = j_M \circ B(\xi,\eta) - B(j_L\xi,\eta) \tag{6}$$

for some B. The lemma states that the map $B : T_xL \otimes T_xL \to T_yM$ might be chosen symmetric. Let us consider the decomposition of B into the symmetric and antisymmetric components:

$$B = B_0 + B_1,\ \ B_i(\xi,\eta) = \frac{1}{2}[B(\xi,\eta) + (-1)^i B(\eta,\xi)].$$

The substitution of this expression into (5) taking into account (6) gives:

$$j_M \circ B_1(\xi,\eta) - B_1(j_L\xi,\eta) - B_1(\xi,j_L\eta) - j_M \circ B_1(j_L\xi,j_L\eta) = 0. \tag{7}$$

Let us decompose B_1 into linear and antilinear by the first and the second argument components:

$$\begin{aligned}
B_1 &= B_1^{0,0} + B_1^{0,1} + B_1^{1,0} + B_1^{1,1},\ \ B_1^{1,1} \equiv 0,\\
B_1^{0,0}(\xi,\eta) &= \frac{1}{2}[B_1(\xi,\eta) - B_1(j_L\xi, j_L\eta)],\\
B_1^{0,1}(\xi,\eta) &= \frac{1}{2}[B_1(\xi,\eta) + j_M \circ B_1(\xi, j_L\eta)],\\
B_1^{1,0}(\xi,\eta) &= \frac{1}{2}[B_1(\xi,\eta) + j_M \circ B_1(j_L\xi, \eta)].
\end{aligned}$$

Using (7) one could easily verify the properties

$$B_1^{r,s}(\xi,\eta) = (-1)^{r+1} j_M \circ B_1^{r,s}(j_L\xi,\eta) = (-1)^{s+1} j_M \circ B_1^{r,s}(\xi, j_L\eta).$$

Note that the tensor

$$\hat{B}_0(\xi,\eta) = B_1^{0,1}(\xi,\eta) - B_1^{1,0}(\xi,\eta) = \frac{j_M}{2}[B_1(\xi, j_L\eta) - B_1(j_L\xi,\eta)]$$

is symmetric and the tensor $B_1 + \hat{B}_0 = B_1^{0,0} + 2B_1^{0,1}$ is (j_L-j_M-)linear by ξ. Therefore the transformation

$$B = B_0 + B_1 \mapsto B - (B_1 + \hat{B}_0) = B_0 - \hat{B}_0$$

determines the required symmetric tensor $\Phi^{(2)} = B_0 - \hat{B}_0$, which satisfies (2). $\square$

Thus the equation $\mathcal{E}^2$ is singled out as a subequation in $\mathcal{E}^1$ by relations (3)-(5). At that the equation (4) gives the identity according to

$$j_M \circ \Phi = \Phi \circ j_L,\ \ j_L \circ dj_L = -dj_L \circ (j_L \otimes \mathbf{1}),\ \ j_M \circ dj_M = -dj_M \circ (j_M \otimes \mathbf{1}). \quad (8)$$

Under substitution of equation (3) into (5) we get

$$\begin{aligned}
&dj_M(\Phi\xi,\Phi\eta) - dj_M(\Phi\eta,\Phi\xi) - dj_M(j_M\Phi\xi, j_M\Phi\eta) + dj_M(j_M\Phi\eta, j_M\Phi\xi) =\\
&\qquad \Phi\left(dj_L(\xi,\eta) - dj_L(\eta,\xi) - dj_L(j_L\xi, j_L\eta) + dj_L(j_L\eta, j_L\xi)\right).
\end{aligned}$$

Multiplying this equality by j_M and using (8) and the identity

$$N_j(\xi,\eta) = -dj(j\xi,\eta) - dj(\xi, j\eta) + dj(j\eta,\xi) + dj(\eta, j\xi), \quad (9)$$

we get the equation $N_{j_M} \circ \Phi^{\wedge 2} = \Phi \circ N_{j_L}$, and the theorem is proved. $\square$

Thus the theorem gives the equations at a point (on T_xL) determining the possibility of pseudoholomorphic mapping construction on 2-jet level.

Let us introduce the linear space of Nijenhuis tensors at the complex linear space $(\mathbb{C}^n, j)$ (further it will be shown in theorem 5 (chapter 4) that every tensor from this space can be realized as the Nijenhuis tensor of some almost complex structure):

$$\mathcal{N}_j^{(n)} = \{N \in \wedge^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n \mid N(j\xi, \eta) = N(\xi, j\eta) = -jN(\xi, \eta)\}. \qquad (10)$$

Consider on $\mathcal{N}_j^{(n)}$ the factorization mapping

$$\theta : \mathcal{N}_j^{(n)} \to \mathcal{N}_j^{(n)} / \operatorname{Gl}_{\mathbb{C}}(n), \qquad (11)$$

which associate to each tensor its orbit under the full group of complex linear transformations. The Cartan's approach to formal integrability (and also one of Spencer, see Goldshmidt's theorem in [KS], [ALV]) assumes to find by the prolongations-projections method the equation $\mathcal{E}^{(\infty)} \subset \mathcal{E}^2 \subset J^1_{PH}$, which prolongs to the 2-jet space and the symbols of which form a bundle. In particular, one needs to require that $(\mathcal{E}^2)_{x',y'} \neq 0$ for all $x' \in L$, $y' \in M$. For example, it is so if one requires the constancy of the natural mappings $\theta_L^{(N)} : \mathcal{O}_L(x) \to \mathcal{N}_{j_L}^{(l)} / \operatorname{Gl}_{\mathbb{C}}(l)$, $\theta_M^{(N)} : \mathcal{O}_M(y) \to \mathcal{N}_{j_M}^{(m)} / \operatorname{Gl}_{\mathbb{C}}(m)$, associating to a point x the image of the Nijenhuis tensor $(N_j)_x$ at this point upon the mapping (11), and the existence of a morphism $\Phi : T_xL \to T_yM$, which conjugates N_{j_L} with N_{j_M}. Manifolds (L, j_L) with $\theta_L^{(N)} \equiv \text{const}$ do exist as the following example shows. Note also the complexity of the linear classification problem for linear Nijenhuis tensors $N \in \mathcal{N}_j^{(n)}$. This problem contains Lie algebras classificational problem as a subproblem.

Example 1. Let's consider a Lie group G. The Lie algebra of the Lie group $G \times G$ is isomorphic to $\mathcal{G} \oplus \mathcal{G}$. Let's introduce the almost complex structure on $T_e(G \times G)$ by formula $j(\xi, \eta) = (-\eta, \xi)$ and extend it by the left shifts onto the whole group. The Nijenhuis tensor is left-invariant and has the form:

$$N((\xi, 0), (\eta, 0)) = -N((0, \xi), (0, \eta)) = (-[\xi, \eta], [\xi, \eta]),$$

$$N((\xi, 0), (0, \eta)) = N((0, \xi), (\eta, 0)) = ([\xi, \eta], [\xi, \eta]). \qquad \square$$

Remark 1. In the paper [Gau] it is shown that there exists a canonical almost complex structure $j^{[1]}$ on the manifold $J^1_{PH}(L,M)$. This structure is defined as $(j_L \oplus j_M) \oplus j_M$ on $T_\bullet J^1_{PH} = T_\bullet J^0_{PH} \oplus Smbl^1_{\mathcal{E}^1}$ in the decomposition induced by minimal connections on manifolds L and M (a minimal connection is by definition such that its torsion form coincides with the torsion form of the almost complex structure, i.e. with the corresponding Nijenhuis tensor, see [L]). At the points $p \in \mathcal{E}^2 \subset J^1_{PH}$ the almost complex structure $j^{[1]}$ on $T_p J^1_{PH}$ may be defined easier. Actually, 1-PH-jet p prolongs to 2-PH-jet $p^{(2)}$, and $p^{(2)}$ determines a subspace $L_{p^{(2)}}$ in the tangent space to J^1_{PH}, i.e. the tangent space to the graphic inducing $p^{(2)}$, cf. [KLV]. Thus, the tangent space $T_p J^1_{PH}$ is isomorphic to the direct sum of the space $Smbl^1_{\mathcal{E}^1}$ — the tangent space to the linear space-fiber and the space $L_{p^{(2)}}$, and since both summand are complex, there exists a canonical complex structure on $T_p J^1_{PH}$. Note that even though the prolongation $p^{(2)}$ may be chosen not in unique way the structure $j^{[1]}_p$ does not depend on it. Suppose that the subsets $(\mathcal{E}^2)_{x',y'}$ do not depend on points $x' \in L$, $y' \in M$ (compare with the above arguments) and form a bundle over J^0_{PH}, i.e. $\mathcal{E}^2$ is a submanifold in $\mathcal{E}^1$. It follows from theorem 1 that this submanifold is not, in general case, complex, and its difference from complex one "measures" the nonintegrability of the structures j_L and j_M. Actually, due to the antilinearity, the quadric determined in the complex linear space $\{\Phi \in T^*_x L \otimes T_y M \mid j_M \circ \Phi = \Phi \circ j_L\}$ by the condition $\{N_{j_M} \circ \Phi^{\wedge 2} = \Phi \circ N_{j_L}\}$ is not a complex submanifold iff $(N_{j_M})|_{\mathrm{Im}\,\Phi^{\wedge 2}} \not\equiv 0$. Hence on J^2_{PH} does not exist a complex structure such that the projection restriction $\pi^{2,1}_{PH} : J^2_{PH} \to J^1_{PH}$ is pseudoholomorphic.

As long as $J^1_{PH}(L,M)$ possesses the canonical almost complex structure $j^{[1]}_{L,M}$, we can consider the manifold $J^{[2]}_{PH}(L,M) = J^1_{PH}(L, J^1_{PH}(L,M))$ with the almost complex structure $j^{[2]}_{L,M}$, and define by induction the manifolds $J^{[k+1]}_{PH}(L,M) = J^1_{PH}(L, J^{[k]}_{PH}(L,M))$ with almost complex structures $j^{[k+1]}_{L,M}$. Note that $J^k_{PH}(L,M) \subset J^{[k]}_{PH}(L,M)$. We say that 1-jet $\Phi \in (J^1_{PH}(L,M))_{x,y}$ is lifted up to the k-jet if $(J^k_{PH})_{x,y} \supset (\pi^{k,1}_{PH})^{-1}(\Phi) \neq \emptyset$. According to theorem 1, 1-jet Φ is lifted up to a 2-jet iff $N_{j_M} \circ \Phi^{\wedge 2} = \Phi \circ N_{j_L}$, and in this case there exists (not necessarily unique) a lift $\Phi^{[2]} \in (J^2_{PH})_{x,y} \subset (J^{[2]}_{PH})_{x,y}$, which could be considered as a 1-jet in the ambient space. This 1-jet can be lifted up to a 2-jet $\Phi^{[3]} \in (J^{[3]}_{PH})_{x,y}$ iff $N_{j^{[1]}_{L,M}} \circ (\Phi^{[2]})^{\wedge 2} = \Phi^{[2]} \circ N_{j_L}$. In general case, if among lifts $\Phi^{[k]} \in (J^{[k]}_{PH})_{x,y}$ of the jet $\Phi^{[k-1]} \in (J^{[k-1]}_{PH})_{x,y}$ there exists such that

$$N_{j^{[k-1]}_{L,M}} \circ (\Phi^{[k]})^{\wedge 2} = \Phi^{[k]} \circ N_{j_L}, \tag{12}$$

then the jet $\Phi^{[k]} \in (J^{[k]}_{PH})_{x,y}$ may be lifted up to a jet $\Phi^{[k+1]} \in (J^{[k+1]}_{PH})_{x,y}$. Thus we have a sequence of necessary conditions (12), and upon their fulfillment there exist lifts $\Phi^{[r]} \in (J^{[r]}_{PH})_{x,y}$ of 1-jet $\Phi \in (J^1_{PH})_{x,y}$, and one can try to find such lifts that $\Phi^{[r]} \in (J^r_{PH})_{x,y}$. $\square$

Chapter 2

The pseudoholomorphic mapping construction on a higher jet level

In chapter 1 we found an algebra of invariants $\mathcal{A}_j^2 =< j, N_j >$ generated relative to commutator by the elements j, N_j, and such that $\mathcal{E}^2_{x,y} = \{\Phi : T_xL \to T_yM \mid \mathcal{A}^2_{j_M} \circ \Phi^{\wedge\star} = \Phi \circ \mathcal{A}^2_{j_L}\}$. Applying the prolongation-projection method (see chapter 1) we contract the equation $\mathcal{E}^2$ onto the equation $\mathcal{E}^\infty$ and find an algebra of invariants $\mathcal{A}_j^\infty$ such that $\mathcal{E}^\infty_{x,y} = \{\Phi : T_xL \to T_yM \mid \mathcal{A}^\infty_{j_M} \circ \Phi^{\wedge\star} = \Phi \circ \mathcal{A}^\infty_{j_L}\}$. As it is shown in remark 7 (section 6.2) neither the algebra $\mathcal{A}_j^2$ nor its closure under differential Nijenhuis bracket (see [FN], [ALV]) — *the Nijenhuis algebra* $\mathcal{A}_j^{\mathcal{N}}$ — coincides with the algebra $\mathcal{A}_j^\infty$ looked for, i.e. $\mathcal{A}_j^{\mathcal{N}} \neq \mathcal{A}_j^\infty =< j, N_j, \dots >$. Below we construct an invariant coming from a higher jet space and being a new (the third) generator of $\mathcal{A}_j^\infty$. Note that performing further computations one could find other generators.

Theorem 2. *There exist new tensor invariants — higher Nijenhuis tensors — such that* $\mathcal{A}_j^\infty =< j, N_j, N_j^{(2)}, \dots >$. *The first of such invariants* $N_j^{(2)}$ *has the following form*

$$\begin{aligned}
&N_j^{(2)}(\xi,\eta,\zeta,\nu) = -[N_j(\xi,\eta), jN_j(\zeta,\nu)] - [jN_j(\xi,\eta), N_j(\zeta,\nu)]\\
&+N_j([\xi, jN_j(\zeta,\nu)],\eta) + N_j(\xi,[\eta, jN_j(\zeta,\nu)]) + jN_j([\xi, N_j(\zeta,\nu)],\eta)\\
&+jN_j(\xi,[\eta, N_j(\zeta,\nu)]) - N_j([\zeta, jN_j(\xi,\eta)],\nu) - N_j(\zeta,[\nu, jN_j(\xi,\eta)])\\
&-jN_j([\zeta, N_j(\xi,\eta)],\nu) - jN_j(\zeta,[\nu, N_j(\xi,\eta)]),
\end{aligned} \tag{13}$$

where ξ, η, ζ, ν are arbitrary vector fields extending the vectors prescribed at the point x. In general case this invariant is the first among generating invariants from $\mathcal{A}_j^\infty$ distinguishing structures j and $(-j)$ (see remark 4).

Proof. As it is shown in theorem 3 for the equation $\tilde{\mathcal{E}}^1 = \mathcal{E}^2 \subset J^1_{PH}$ the projections $\tilde{\mathcal{E}}^k = \mathcal{E}^{k+1}$. So instead of consideration the problem on projections of higher prolongations of the equation $\mathcal{E}^1$, which leads to a formally integrable equation (see chapter 1), one may consider the problem of finding out the projections of higher prolongations of the new equation $\tilde{\mathcal{E}}^1$, and at the first step, the problem of lifting on one level. The solution on 1-jet level of the equation $\tilde{\mathcal{E}}^1$ prolongs to the solution on 2-jet level iff there holds equality (1) and:

$$\begin{aligned} U(\xi,\eta,\theta) :&= \Phi^{(2)}(N_{j_L}(\xi,\eta),\theta) - N_{j_M}(\Phi^{(2)}(\xi,\theta),\Phi\eta) - N_{j_M}(\Phi\xi,\Phi^{(2)}(\eta,\theta)) \\ &= d(N_{j_M})(\Phi\xi,\Phi\eta,\Phi\theta) - \Phi\circ d(N_{j_L})(\xi,\eta,\theta). \end{aligned} \tag{14}$$

From (1) and (14) we have:

$$\begin{aligned} &U(\xi,\eta,j_L\theta) + j_M U(\xi,\eta,\theta) = 2j_M\circ\Phi^{(2)}(N_{j_L}(\xi,\eta),\theta) \\ &\quad + dj_M(\Phi\theta, N_{j_M}(\Phi\xi,\Phi\eta)) - \Phi\circ dj_L(\theta, N_{j_L}(\xi,\eta)) - N_{j_M}(dj_M(\Phi\theta,\Phi\xi),\Phi\eta) \\ &\quad - N_{j_M}(\Phi\xi, dj_M(\Phi\theta,\Phi\eta)) + \Phi\circ N_{j_L}(dj_L(\theta,\xi),\eta) + \Phi\circ N_{j_L}(\xi, dj_L(\theta,\eta)) \\ &= d(N_{j_M})(\Phi\xi,\Phi\eta,j_M\Phi\theta) + j_M\circ d(N_{j_M})(\Phi\xi,\Phi\eta,\Phi\theta) \\ &\quad - \Phi\circ d(N_{j_L})(\xi,\eta,j_L\theta) - \Phi\circ j_L\circ d(N_{j_L})(\xi,\eta,\theta). \end{aligned}$$

Hence:

$$2j_M\circ\Phi^{(2)}(N_j(\xi,\eta),\theta) = R_{j_M}(\Phi\xi,\Phi\eta,\Phi\theta) - \Phi\circ R_{j_L}(\xi,\eta,\theta),$$

where

$$\begin{aligned} R_j(\xi,\eta,\theta) &= d(N_j)(\xi,\eta,j\theta) + j d(N_j)(\xi,\eta,\theta) \\ &\quad + N_j(dj(\theta,\xi),\eta) + N_j(\xi, dj(\theta,\eta)) - dj(\theta, N_j(\xi,\eta)). \end{aligned} \tag{15}$$

Since $\Phi^{(2)}$ is a symmetric tensor then for the tensor

$$N_j^{(2)}(\xi,\eta,\zeta,\nu) = R_j(\xi,\eta,N_j(\zeta,\nu)) - R_j(\zeta,\nu,N_j(\xi,\eta)) \tag{16}$$

we have:

$$N^{(2)}_{j_M} \circ (\Phi^{\wedge 2})^{\wedge 2} = \Phi \circ N^{(2)}_{j_L}. \qquad (17)$$

Therefore according to (17) we have a new invariant, which we will call *a higher Nijenhuis tensor*, $N^{(2)}_j \in \mathcal{A}^3_j \subset \mathcal{A}^\infty_j$. Formula (16) shows the tensor nature of the invariant $N^{(2)}_j \in \wedge^2(\wedge^2 T^*_x) \otimes T_x$, while formula (13), following from formulae (14)-(15), shows its independence of the Euclidean connection ∇ which defines $d = d_\nabla$. Note that a priori the tensor of the valence (1,4) $N^{(2)}_j$ might become dependent with tensors of forth order coming from $\mathcal{A}^2_j$ such as $N_j(N_j(\xi,\eta), N_j(\zeta,\nu))$, i.e. not to be new, but example 2 shows that this is not the case. □

Example 2. Let $\partial_i = \dfrac{\partial}{\partial x^i}$ be basis vectors on $\mathbb{R}^4 = \mathbb{R}^4(x^1, x^2, x^3, x^4)$. Let's define an almost complex structure by the formula:

$$j\partial_1 = \partial_2,\ j\partial_2 = -\partial_1,\ j\partial_3 = \partial_4 + x^2\partial_1,\ j\partial_4 = -\partial_3 - x^2\partial_2.$$

For the tensors N_j and $N^{(2)}_j$ we have:

$$N_j(\partial_1, \partial_2) = 0, \quad N_j(\partial_3, \partial_4) = x^2\partial_1,$$

$$N_j(\partial_1, \partial_3) = -N_j(\partial_2, \partial_4) = \partial_1,\ N_j(\partial_2, \partial_3) = N_j(\partial_1, \partial_4) = -\partial_2;$$

$$N^{(2)}_j(\partial_1, \partial_3, \partial_3, \partial_4) = -\partial_1,$$
$$N^{(2)}_j(\partial_1, \partial_3, j\partial_3, \partial_4) = N^{(2)}_j(\partial_1, \partial_3, \partial_4, \partial_4) + x^2 N^{(2)}_j(\partial_1, \partial_3, \partial_1, \partial_4) = 0. \quad \square$$

Chapter 3

A solvability criterion for the Cauchy-Riemann equation and formal normal forms of almost complex structures

3.1 Solvability of the Cauchy-Riemann equation

The set of PH-jets may be locally described as follows:

$$
\begin{aligned}
J_{PH}^k = \{(x, y, \Phi^{(1)}, \dots, \Phi^{(k)}) \mid\ & x \in L, y \in M, \\
& \Phi^{(i)} \in Smbl_{PH}^i(L, M) \subset S^i T^* L \otimes TM\},
\end{aligned} \tag{18}
$$

$$
\sum_{\substack{r_1+\ldots+r_p=k \\ 1\le p\le k}} \quad \sum_{\substack{1\le i_s^r\le k, \#\{i_s^r\}=k \\ i_1^1<i_2^1<\ldots<i_p^1 \\ i_s^1<i_s^2<\ldots<i_s^{r_s}}} d^{p-1} j_M \left(\Phi^{(r_1)}(\xi_{i_1^1}, \dots, \xi_{i_1^{r_1}}), \dots, \Phi^{(r_p)}(\xi_{i_p^1}, \dots, \xi_{i_p^{r_p}})\right) =
$$

$$
\sum_{\substack{1\le p\le k, i_1^1=1, 1\le i_s^r\le k, \\ \#\{i_s^r\}=k, i_s^1<i_s^2<\ldots<i_s^*}} \Phi^{(k-p+1)} \left(dj_L^{p-1}(\xi_{i_1^1}, \dots, \xi_{i_1^p}), \xi_{i_2^1}, \dots, \xi_{i_2^{k-p}}\right).
$$

Here $\#S$ denotes the number of different elements of the set S.

The following theorem generalizes theorem 1 corresponding to the case $k = 2$.

Theorem 3. *Let the mapping* $u : \mathcal{O}_L(x) \to \mathcal{O}_M(y)$ *be such that*

$$j_M \circ u_* = u_* \circ j_L (\operatorname{mod} \mu^{k-1}),\ N_{j_M} \circ u_*^{\wedge 2} \equiv u_* \circ N_{j_L} (\operatorname{mod} \mu^{k-1})\ (k \geq 2). \quad (19)$$

Then there exists a mapping $\tilde{u} : \mathcal{O}_L(x) \to \mathcal{O}_M(y)$ *such that*

$$j_M \circ \tilde{u}_* - \tilde{u}_* \circ j_L \in \mu^k,\ u_* - \tilde{u}_* \in \mu^{k-1}.$$

In particular if $(u_*)_x \neq 0$ *then* $(\tilde{u}_*)_x \neq 0$*, and if* $(u_*)_x$ *is an embedding or a surjection then the same is true for* $(\tilde{u}_*)_x$.

Proof. A $(k-1)$-PH-jet defined by means of symbols $(\Phi, \Phi^{(2)}, \ldots, \Phi^{(k-1)})$ is lifted up to k-PH-jet iff some symmetric tensor $\Phi^{(k)}$ satisfies equation (18). Consider the tensor

$$P_k = j_M \circ \Phi^{(k)} - \Phi^{(k)} \circ (j_L \otimes \mathbf{1}_L^{k-1}). \quad (20)$$

Lemma $\mathbf{1_k}$. (i_k) $\quad j_M \circ P_k = -P_k \circ (j_L \otimes \mathbf{1}_L^{k-1})$,
(ii_k) $\ P_k(\xi, \eta, \vec{\theta}^{k-2}) - P_k(\eta, \xi, \vec{\theta}^{k-2}) = P_k(j_L\xi, j_L\eta, \vec{\theta}^{k-2}) - P_k(j_L\eta, j_L\xi, \vec{\theta}^{k-2})$,
(iii_k) $P_k \in T_x^*L \otimes S^{k-1}T_x^*L \otimes T_yM$. $\square$

Lemma $\mathbf{2_k}$. Every tensor P_k, satisfying the conditions of lemma 1_k may be represented in form (20), where $\Phi^{(k)} \in S^kT_x^*L \otimes T_yM$.

Proof of lemma 2_k. Let's consider the tensor $B(\xi, \vec{\eta}^{(k-1)})$ such that $P_k = j_M \circ B - B \circ (j_L \otimes \mathbf{1}_L^{k-1})$. Due to (i_k) it always can be found, and because of (iii_k) one can suppose that B is symmetric by the last $(k-1)$ arguments: $B \in T_x^*L \otimes S^{k-1}T_x^*L \otimes T_yM$. Note that under the change of B by a tensor j_L-j_M-linear by the first argument, the corresponding tensor P_k does not change. Thus in the decomposition $B = B^{0,*} + B^{1,*}$ by j_L-j_M-linear and antilinear by the first argument components one may eliminate the first term and assume that $B = B^{1,*}$.

Consider the decomposition

$$B^{1,*} = \sum_{p=0}^{k-1} C_p,$$

where $C_p = \sum\limits_{\tau_p \in \mathbb{Z}_2^{k-1}} B^{1,\tau_p}$, and $\mathbb{Z}_2 = \{0,1\}$, the number of units in τ_p is equal to $\#_1\tau_p = p$, and the tensor B^{1,τ_p} is a j_L-j_M-linear or antilinear by s-th argument if there's placed, respectively, 0 or 1 on the s-th place in $(1,\tau_p)$. Note that if units form the set $i_1,\dots,i_r$ in τ_p and zeros form the set $j_1,\dots,j_{k-r-1}$, then the tensor B^{1,τ_p} is symmetric by the corresponding arguments $\xi_{i_1},\dots,\xi_{i_r}$ and by $\xi_{j_1},\dots,\xi_{j_{k-r-1}}$. Note also that the tensor B^{1,τ_p} is symmetric by the arguments ξ_1,ξ_{i_s}. Actually, for example, for $B^{1,1,*}$ from the formula (ii_k) we have (we use j_L-j_M-antilinearity by ξ_1):

$$B^{1,*}(\xi,\eta,\vec{\theta}^{k-2})+j_M B^{1,*}(\xi,j_L\eta,\vec{\theta}^{k-2}) = B^{1,*}(\eta,\xi,\vec{\theta}^{k-2})+j_M B^{1,*}(\eta,j_L\xi,\vec{\theta}^{k-2}),$$

$$\text{i.e.}\quad B^{1,1,*}(\xi,\eta,\vec{\theta}^{k-2}) = B^{1,1,*}(\eta,\xi,\vec{\theta}^{k-2}).$$

Thus the tensor B^{1,τ_p} is symmetric by arguments $\xi_1,\xi_{i_1},\dots,\xi_{i_r}$. It is not hard to show that if $\gamma \in S_{k-1}$ is a permutation then the tensor $B^{1,\gamma(\tau_p)}$ is equal to $B^{1,\tau_p}\circ(\mathbf{1}\times\gamma)$, where $(\mathbf{1}\times\gamma)$ acts by permutation on arguments; in the case when $\gamma(\tau_p)=\tau_p$ the statement coincides with the noted above symmetry property of B^{1,τ_p}. Hence if we add the sum $\sum\limits_{\#_1\tau_{p+1}=p+1} B^{0,\tau_{p+1}}$ to the tensor C_p, where $B^{0,\tau_{p+1}} = B^{1,\tau_p}\circ\sigma$ for $\sigma\in S_k$ being such a permutation that $(0,\tau_{p+1}) = \sigma^{-1}(1,\tau_p)$, then we obtain a new symmetric tensor $\hat{C}_p \in S^kT_x^*L\otimes T_yM$.

Let us consider the tensor $\Phi^{(k)} = \sum\limits_{p=0}^{k-1}\hat{C}_p$. It is symmetric by the construction and $(\Phi^{(k)}-B)$ is j_L-j_M-linear. Therefore the tensor $P_k = j_M\circ\Phi^{(k)} - \Phi^{(k)}\circ(j_L\otimes\mathbf{1}^{k-1})$ constructed by $\Phi^{(k)}$ is equal to the similar tensor constructed by B, and lemma 2_k is proved.

Note that the proposed construction slightly differs from the construction of lemma 2 ($k=2$) in theorem 1. For the case $k=3$ we have:

$$B^{1,*} = B^{1,0,0} + (B^{1,0,1}+B^{1,1,0}) + B^{1,1,1}.$$

The tensor $B^{1,1,1}$ is symmetric, and $B^{1,0,0}$ is symmetric by the last two arguments (η,θ). Hence if σ is a cyclic permutation of arguments, $\sigma(\xi,\eta,\theta) = (\eta,\theta,\xi)$, then $B^{1,0,0}\circ\sigma$ and $B^{1,0,0}\circ\sigma^2$ are j_L-j_M-linear by the first argument, and the tensor $(B^{1,0,0}+B^{1,0,0}\circ\sigma+B^{1,0,0}\circ\sigma^2)$ is symmetric. The tensor $B^{1,1,0}$ is symmetric by the arguments (ξ,η) due to (ii_3). So $B^{1,1,0}(\xi,\eta,\theta) =$

$A(\xi,\eta,\theta)+A(\eta,\xi,\theta)$ for some tensor A j_L-j_M-antilinear by the first two arguments and linear by the last. Moreover, the tensor $B^{1,*}$ is invariant under the permutation τ of the last two arguments, $\tau(\xi,\eta,\theta)=(\xi,\theta,\eta)$. Therefore

$$\begin{aligned} B^{1,0,1}(\xi,\eta,\theta) &= B^{1,1,0}\circ\tau(\xi,\eta,\theta)=A(\xi,\theta,\eta)+A(\theta,\xi,\eta),\\ B^{1,0,1}(\xi,\eta,\theta) &= B^{1,1,0}(\xi,\theta,\eta)=B^{1,1,0}(\theta,\xi,\eta)=B^{1,1,0}\circ\sigma^2(\xi,\eta,\theta). \end{aligned}$$

Let's define the tensor

$$B^{0,1,1}(\xi,\eta,\theta):=B^{1,1,0}\circ\sigma(\xi,\eta,\theta)=A(\eta,\theta,\xi)+A(\theta,\eta,\xi).$$

Note that $(B^{1,1,0}+B^{1,0,1}+B^{0,1,1})$ is a symmetric tensor and $B^{0,1,1}$ is j_L-j_M-linear by the first argument. So the tensor

$$P_k=j_M\circ B^{1,*}-B^{1,*}\circ(j_L\otimes\mathbf{1}\otimes\mathbf{1})=j_M\circ\Phi^{(3)}-\Phi^{(3)}\circ(j_L\otimes\mathbf{1}\otimes\mathbf{1}),$$

where $\Phi^{(3)}\in S^3T_x^*L\otimes T_yM$ is defined by the equality

$$\Phi^{(3)}=(B^{1,0,0}+B^{1,0,0}\circ\sigma+B^{1,0,0}\circ\sigma^2)+(B^{1,1,0}+B^{1,0,1}+B^{0,1,1})+B^{1,1,1}.$$

$\square$

The statement of the lemmas may be reformulated as exactness of the following complex:

$$0\to\mathcal{G}_k\xrightarrow{i}S^kT^*L\otimes TM\xrightarrow{\sigma_k}T^*L\otimes S^{k-1}T^*L\otimes TM\xrightarrow{\kappa_k\oplus\nu_k}(\Theta_k\oplus\Delta_k)\to 0.$$

Here $\mathcal{G}_k$ is the prolongation of the symbol ([ALV], [KLV]) of the Cauchy-Riemann equation $\mathcal{E}^1$, $S^kT^*L\otimes TM$ is the space of all the symbols, i is the inclusion, and σ_k is the linearization of the operator $\bar\partial$:

$$\sigma_k(\Psi)=j_M\circ\Psi-\Psi\circ(j_L\otimes\mathbf{1}^{k-1}).$$

The maps κ_k and ν_k on the spaces

$$\Theta_k=\wedge^2T^*L\otimes S^{k-2}T^*L\otimes TM \text{ and } \Delta_k=S^{k-1}T^*L\otimes\mathrm{Hom}_{\mathbf{C}}(TL,TM)$$

have the form:

$$\kappa_k=\delta_k\circ\tilde\sigma_k \text{ and } \nu_k(X\otimes\vec{Y}^{(k-1)}\otimes Z)=(\vec{Y}^{(k-1)}\otimes\epsilon(X\otimes Z)),$$

where δ_k is the operator of alternation by the first two arguments,

$$\delta_k(B)(X, Y, \vec{Z}^{(k-2)}) = B(X, Y, \vec{Z}^{(k-2)}) - B(Y, X, \vec{Z}^{(k-2)}),$$

$$\tilde{\sigma}_k \Psi = j_M \circ \Psi - \Psi \circ (\mathbf{1} \otimes j_L \otimes \mathbf{1}^{k-2}),$$

and $\epsilon(B) = j_M \circ B + B \circ j_L$ is the linearization of the operator ∂. Note that the term Θ_k, corresponding to the condition (ii_k) of lemma 1_k, gives the preserving condition for the Nijenhuis tensor differential and the term Δ_k, corresponding to the condition (i_k), gives the identity. In more details, let us express the tensor P_k by local representative $(\Phi, \Phi^{(2)}, \ldots, \Phi^{(k-1)})$ of $(k-1)$-jet of the mapping $u : \mathcal{O}_L(x) \to \mathcal{O}_M(y)$ by means of formula (18) and let us substitute the corresponding expression into the formulae of lemma 1_k. Then the conditions (i_k) and (iii_k) give identities and (ii_k) gives the preserving condition for $(k-1)$-jet of the Nijenhuis tensor. The relevant calculations are quite similar to those of theorem 1 which correspond to the case $k = 2$.

For example, if $k = 3$ we have:

$$\begin{aligned}
&\Phi_3(\xi, \eta, \theta) = j_M \Phi^{(3)}(\xi, \eta, \theta) - \Phi^{(3)}(j_L \xi, \eta, \theta) \\
&= \Phi d^2 j_L(\xi, \eta, \theta) - d^2 j_M(\Phi\xi, \Phi\eta, \Phi\theta) + \Phi^{(2)}(dj_L(\xi, \theta), \eta) + \Phi^{(2)}(dj_L(\xi, \eta), \theta) \\
&- dj_M(\Phi^{(2)}(\xi, \theta), \Phi\eta) - dj_M(\Phi\xi, \Phi^{(2)}(\eta, \theta)) - dj_M(\Phi^{(2)}(\xi, \eta), \Phi\theta). \qquad (21)
\end{aligned}$$

The substitution of expression (21) into (i_3) and (iii_3) gives the identity according to formulae (8) and the formula

$$d^2 j(j\xi, \eta, \theta) = -j d^2 j(\xi, \eta, \theta) - dj(dj(\xi, \theta), \eta) - dj(dj(\xi, \eta), \theta). \qquad (22)$$

Let us substitute expression (21) into (ii_3) and use formulae (1), (8), (22) and the identity

$$\begin{aligned}
&dj(N_j(\xi, \eta), \theta) + j dN_j(\xi, \eta, \theta) = \\
&d^2 j(j\xi, j\eta, \theta) - d^2 j(j\eta, j\xi, \theta) - d^2 j(\xi, \eta, \theta) + d^2 j(\eta, \xi, \theta) + \\
&dj(dj(\xi, \theta), j\eta) - dj(dj(\eta, \theta), j\xi) + dj(j\xi, dj(\eta, \theta)) - dj(j\eta, dj(\xi, \theta)),
\end{aligned}$$

which is obtained from identity (9) by differentiation. We have*:

*We omit a page of calculations

$$\begin{aligned}
0 = &\Phi^{(2)}(dj_L(\xi,\theta),\eta) + \Phi^{(2)}(dj_L(\xi,\eta),\theta)) - \\
&dj_M(\Phi^{(2)}(\xi,\theta),\Phi\eta) - dj_M(\Phi\xi,\Phi^{(2)}(\eta,\theta)) - dj_M(\Phi^{(2)}(\xi,\eta),\Phi\theta) + \\
&\Phi d^2 j_L(\xi,\eta,\theta) - d^2 j_M(\Phi\xi,\Phi\eta,\Phi\theta) - \\
&\Phi^{(2)}(dj_L(\eta,\theta),\xi) - \Phi^{(2)}(dj_L(\eta,\xi),\theta) + \\
&dj_M(\Phi^{(2)}(\eta,\theta),\Phi\xi) + dj_M(\Phi\eta,\Phi^{(2)}(\xi,\theta)) + dj_M(\Phi^{(2)}(\eta,\xi),\Phi\theta) - \\
&\Phi d^2 j_L(\eta,\xi,\theta) + d^2 j_M(\Phi\eta,\Phi\xi,\Phi\theta) - \\
&\Phi^{(2)}(dj_L(j_L\xi,\theta),j_L\eta) - \Phi^{(2)}(dj_L(j_L\xi,j_L\eta),\theta) + \\
&dj_M(\Phi^{(2)}(j_L\xi,\theta),j_M\Phi\eta) + dj_M(j_M\Phi\xi,\Phi^{(2)}(j_L\eta,\theta)) + dj_M(\Phi^{(2)}(j_L\xi,j_L\eta),\Phi\theta) \\
&-\Phi d^2 j_L(j_L\xi,j_L\eta,\theta) + d^2 j_M(j_M\Phi\xi,j_M\Phi\eta,\Phi\theta) + \\
&\Phi^{(2)}(dj_L(j_L\eta,\theta),j_L\xi) + \Phi^{(2)}(dj_L(j_L\eta,j_L\xi),\theta) - \\
&dj_M(\Phi^{(2)}(j_L\eta,\theta),j_M\Phi\xi) - dj_M(j_M\Phi\eta,\Phi^{(2)}(j_L\xi,\theta)) - dj_M(\Phi^{(2)}(j_L\eta,j_L\xi),\Phi\theta) \\
&+\Phi d^2 j_L(j_L\eta,j_L\xi,\theta) - d^2 j_M(j_M\Phi\eta,j_M\Phi\xi,\Phi\theta) \\
= &j_M[-\Phi^{(2)}(N_{j_L}(\xi,\eta),\theta) - \Phi dN_{j_L}(\xi,\eta,\theta) + \\
&N_{j_M}(\Phi^{(2)}(\xi,\theta),\Phi\eta) + N_{j_M}(\Phi\xi,\Phi^{(2)}(\eta,\theta)) + dN_{j_M}(\Phi\xi,\Phi\eta,\Phi\theta)].
\end{aligned}$$

But the vanishing of the last expression means exactly that the 1-jet from $\tilde{\mathcal{E}}^1$ may be prolonged to a 2-jet.

We omit calculations for the case of general k. The theorem is proved. □

Corollary. Let the Nijenhuis tensors be of smallness of order k at the point $x \in L$ and $y \in M$: $N_{j_L} \in \mu_x^k$, $N_{j_M} \in \mu_y^k$ $(1 \le k \le \infty)$. Then there exists such a mapping $u : \mathcal{O}_L(x) \to \mathcal{O}_M(y)$ that $du_x \neq 0$ and

$$j_M \circ u_* - u_* \circ j_L \in \mu^{k+1}. \qquad \square$$

Remark 2. When $\dim_{\mathbb{C}} L = \dim_{\mathbb{C}} M = n$, $k = \infty$, this assertion is the formal part of Newlander-Nirenberg theorem (see [NN], [NW]), stating that if the Nijenhuis tensors of the almost complex structures j and j_0 coincide then these structures are locally isomorphic; here j_0 is the standard complex structure in $\mathcal{O}_{\mathbb{C}^n}(0)$ (certainly $N_{j_0} \equiv 0$, see formula (9)). A generalization of this fact is theorem 6 below. Note also that smooth equivalence does not follows from the formal one. Actually, one may construct an almost complex structure j such that $N_j \in \mu^\infty$, but $N_j \not\equiv 0$ (see the proof of theorem 5 at chapter 4). Then j is formally equivalent to j_0 but not equivalent smoothly. □

Remark 3. The conditions of the corollary might be reformulated in the following form: $dN_{j_L} \in \mu_x^{k-1}$, $dN_{j_M} \in \mu_y^{k-1}$, where $d = d_\nabla$ and ∇ is a trivial connection. Let us note that there are no almost complex structures save complex ones such that $d(N_j) \equiv 0$. Actually, in this case there exist coordinates in which the components of the Nijenhuis tensors are constant, and additionally we may suppose that the induced coordinates in the fixed tangent space $T_x \simeq \mathbb{C}^n$ are complex. Let's write the action of the almost complex structure on the basis vectors $\partial_r = \frac{\partial}{\partial x^r}$:

$$j\partial_{2r-1} = \partial_{2r} + \sum a_{2r}^i \partial_i, \; j\partial_{2r} = -\partial_{2r-1} + \sum a_{2r-1}^i \partial_i; \quad a_t^s \in \mu_x.$$

If for almost any vector ξ: $\dim N_j(\xi, T_x) = 2n-2$, i.e. the tensor N_j is of general position, $\Im_{N_j} = 2$ almost everywhere, in the sense of definition 4 from chapter 4, then from $N_j(j\partial_{2r-1}, \partial_{2r}) \equiv N_j(j\partial_{2s-1}, \partial_{2s}) \equiv 0$ it follows that $a_{2s}^{2r} \equiv a_{2s-1}^{2r} \equiv a_{2s}^{2r-1} \equiv a_{2s-1}^{2r-1} \equiv 0$ for all $r \neq s$, i.e. by definition $N_j(\partial_{2r}, \partial_{2s}) = 0$; contradiction. Let us suppose now that there is a subspace $K_1 \subset T_x$ ($\dim K_1 \geq 4$) such that $N_j|_{\wedge^2 K_1} \equiv 0$, and we also suppose that K_1 is maximal satisfying this property. Extending K_1 by affine shifts in the fixed coordinate system to a neighborhood of the point x we obtain a foliation $\mathcal{K}_1$, $\dim \mathcal{K}_1 = \dim K_1$, j-invariant according to $N_j(j\xi, \eta) = -jN_j(\xi, \eta) = 0$ for $\xi, \eta \in T_\bullet \mathcal{K}_1$, and the restrictions of j to fibers of which are integrable, $N_j|_{\mathcal{K}_1} \equiv 0$. In an additional subspace let's choose a maximal degenerate relative to N_j subspace K_2, and construct the foliation $\mathcal{K}_2$ by it and so on. Finally, we are lead to the situation when the almost complex structure j in a neighborhood $\mathcal{O}(x)$ is represented as a direct sum of complex structures on the fibers of the standard foliations $\mathcal{K}_i$, which are obtained from the summand of the complex decomposition $T_x = \oplus K_i$. Now it is easy to obtain contradiction with the condition $dN_j \equiv 0$, if j is not globally integrable.

Let us note however that there are nontrivial connections ∇ such that $d_\nabla(N_j) \equiv 0$. Actually, if one takes for ∇ an almost complex connection ([L]), then $d_\nabla(j) \equiv 0$, which implies $d_\nabla(N_j) \equiv 0$. In particular, this property is satisfied by a minimal connection ∇, which is nonsymmetric and have the torsion tensor equal to $T_\nabla = N_j$; such connections always exist, see [L]. $\square$

3.2 Formal normal forms of almost complex structures

By means of theorem 3 it is possible to construct *formal normal forms of almost complex structures*. One may consider almost complex structures *modulo* μ^k, i.e. such automorphisms $j : T_{x'}X \to T_{x'}X$, $x' \in X$, that $j^2 = -\mathbf{1}_X (\mathrm{mod}\ \mu_x^k)$.

Let us suppose that we constructed normal forms of almost complex structures *modulo* μ^k: $\mathcal{J}_k = \{j_{\lambda_1,\dots,\lambda_{k-1}}\}$. Here $\lambda_1, \dots, \lambda_{k-1}$ are parameters determining elements $\mathcal{J}_k$. Consider those of the structures that prolong to almost complex structures *modulo* μ^{k+1}, i.e. those $J_k \in \mathcal{J}_k$ that there exists an automorphism of the tangent bundle $\Delta J_k \in \mu^k$ for which the structure $J_k + \Delta J_k$ is almost complex *modulo* μ^{k+1}; let us call such structures k-compatible. If J_k is almost complex structure *modulo* μ^{k+1} and $\Delta J_k \in \mu^k$, then $J_k + \Delta J_k$ is almost complex structure *modulo* μ^{k+1} iff $j_0 \circ \Delta J_k + \Delta J_k \circ j_0 \in \mu^{k+1}$, where $j_0 = J_k (\mathrm{mod}\, \mu)$ is the linearization of the structure J_k.

Proposition 1. *Two almost complex modulo* μ^{k+1} *($k \geq 1$) structures* J_k *and* $J_k + \Delta J_k$ *are equivalent by module* μ^{k+1} *iff the automorphism* ΔJ_k *is compatible with the complex structure* j_0 *by module* μ^{k+1}*:*

$$\begin{aligned} N_{(j_0,\Delta J_k)}(\xi,\eta) &:= [j_0\xi, \Delta J_k \eta] + [\Delta J_k \xi, j_0 \eta] - j_0[\xi, \Delta J_k \eta] \\ &- \ j_0[\Delta J_k \xi, \eta] - \Delta J_k[\xi, j_0\eta] - \Delta J_k [j_0 \xi, \eta] \in \mu^{k+1}. \end{aligned}$$

Actually, $N_{(J_k+\Delta J_k, J_k)} := N_{(J_k+\Delta J_k)} - N_{J_k} \equiv N_{(j_0,\Delta J_k)} (\mathrm{mod}\ \mu^{k+1})$. □

Thus, k-compatible almost complex structures $j_{\lambda_1,\dots,\lambda_{k-1}}$ from $\mathcal{J}_k$ define normal forms of almost complex structures *modulo* μ^{k+1} $j_{\lambda_1,\dots,\lambda_k}$ from $\mathcal{J}_{k+1}$, where the parameter λ_k takes values in the symbols of k-jets of the compatibility Nijenhuis tensors

$$\lambda_k = [N_{(j_0,\Delta J_k)}]_k \in (\mu^k \wedge^2 T_x^* \otimes T_x)/(\mu^{k+1} \wedge^2 T_x^* \otimes T_x) = \wedge^2 T_x^* \otimes S^k T_x^* \otimes T_x.$$

(compare the values space with the term Θ_{k+2} in the complex from the proof of theorem 3).

So we have constructed normal forms of almost complex structures *modulo* μ^{k+1}: $\mathcal{J}_{k+1} = \{j_{\lambda_1,\dots,\lambda_k}\}$, i.e. we have found such structures that for every almost complex structure (*modulo* μ^{k+1}) $j : \mathcal{O}(x) \to \mathcal{O}(x)$ there exists a diffeomorphism $\varphi_k : \mathcal{O}(x) \to \mathcal{O}(x)$ and a collection of parameters $\lambda_1, \ldots, \lambda_k$, which satisfy the identity

$$\varphi_k^* j \equiv j_{\lambda_1,\ldots,\lambda_k} (\text{mod } \mu^{k+1}).$$

Let's call *the tree of almost complex structures* the hierarchy $\mathcal{J}_1 = \{j_0\}$, $\mathcal{J}_2 = \{j_{\lambda_1}\}$, $\mathcal{J}_3 = \{j_{\lambda_1,\lambda_2}\}$, and so on. Let's call *a branch of the tree* any sequence of prolonging each other almost complex structures from the tree j_0, j_{λ_1}, j_{λ_1,λ_2}, $j_{\lambda_1,\lambda_2,\lambda_3}$, $\ldots$; let us denote such a sequence by the symbol $j_{\{\lambda_i\}_1^\infty}$, and their union by the symbol $\mathcal{J}_\infty$. By Borel's theorem for any formal diffeomorphism there exists a smooth one inducing it. Hence from the arguments above it follows the statement:

Theorem 4. *The set $\mathcal{J}_\infty$ defines formal normal forms of almost complex structures modulo μ^∞, i.e. for any formal almost complex structure j, $j^2 + \mathbf{1} \in \mu^\infty$, there exists such an element $j_{\{\lambda_i\}_1^\infty} \in \mathcal{J}_\infty$ and a diffeomorphism $\varphi : \mathcal{O}(x) \to \mathcal{O}(x)$, that*

$$\varphi^* j \equiv j_{\{\lambda_i\}_1^\infty} (\text{mod } \mu^\infty). \qquad \square$$

Note that the constructed normal forms are not minimal in the sense of decomposition on nonintersecting classes even *modulo* μ^2, i.e. by module of Nijenhuis tensors classification at a point, see the arguments after theorem 1. But nevertheless this classification essentially reduces the number of possible Taylor series expansions of almost complex structures, e.g. there is only one branch of the tree which correspond to all complex structures.

Chapter 4

Linear theory of Nijenhuis tensors

Let us consider a Nijenhuis tensor $N = N_j$ on a fixed tangent space T_x as a bilinear map $N : T_x \otimes T_x \to T_x$. This map is antilinear and skew-symmetric, $N \in \mathcal{N}_j$, see (10). There're no other constraints on it.

Theorem 5. *For any tensor $N \in \mathcal{N}_{j_0}(T_x)$ (see (10)) there exists an almost complex structure j in $\mathcal{O}(x)$ such that $j_x = j_0$ and $N = (N_j)_x$.*

Proof. Let (x^i) be such a coordinate system in a neighborhood of the point x that the vectors $\partial_i = \left.\dfrac{\partial}{\partial x^i}\right|_x$ form the standard complex relative to j_0 basis in T_x. Then in a neighborhood of $x \in X$:

$$\begin{aligned} j\partial_{2r-1} &= \partial_{2r} + \sum a^i_{2r}\partial_i \\ j\partial_{2r} &= -\partial_{2r-1} + \sum a^i_{2r-1}\partial_i; \quad a^i_s(0) = 0; \end{aligned} \tag{23}$$

we suppose the coordinates $x^i = 0$ at the point x. We have:

$$\begin{aligned} N_j(\partial_{2s-1}, \partial_{2t-1})_0 =& \sum \left(\frac{\partial a^{2i-1}_{2t}}{\partial x^{2s}}(0) - \frac{\partial a^{2i-1}_{2s}}{\partial x^{2t}}(0) + \frac{\partial a^{2i}_{2t}}{\partial x^{2s-1}}(0) - \frac{\partial a^{2i}_{2s}}{\partial x^{2t-1}}(0) \right) \partial_{2i-1} \\ &+ \sum \left(\frac{\partial a^{2i}_{2t}}{\partial x^{2s}}(0) - \frac{\partial a^{2i}_{2s}}{\partial x^{2t}}(0) - \frac{\partial a^{2i-1}_{2t}}{\partial x^{2s-1}}(0) + \frac{\partial a^{2i-1}_{2s}}{\partial x^{2t-1}}(0) \right) \partial_{2i} \\ =& \sum \left(c^{2i-1}_{s,t}\partial_{2i-1} + c^{2i}_{s,t}\partial_{2i} \right). \end{aligned} \tag{24}$$

The theorem states that under arbitrary choice of the constants $c^i_{s,t}$, $1 \le i \le 2n$, $1 \le s < t \le n$, equation (24) has a solution. Actually, let $a^i_{2t}(x^1, \dots, x^{2n}) = \sum\limits_{s<t} c^i_{s,t} x^{2s}$ and let's define a^i_{2t-1} from the condition $j^2 = -\mathbf{1}$, using formula (23). $\square$

Definition 3. Let's call *the space of linear Nijenhuis tensors* the set of tensors $\mathcal{N}^{(n)}_j$ on a fixed complex linear space $V \simeq \mathbb{C}^n$, defined by formula (10).

Let us consider the map associating to a Nijenhuis tensor $N \in \mathcal{N}_j(V)$ on $V \simeq \mathbb{C}^n$ the function $\Im\mathfrak{l}_N : V \to 2\mathbb{Z}_+$:

$$\Im\mathfrak{l}_N(\xi) := \dim\{\eta \mid N(\xi, \eta) = 0\} = 2\dim_{\mathbb{C}}\{\eta \mid N(\xi, \eta) = 0\} \in 2\mathbb{Z}_+.$$

Since $\Im\mathfrak{l}_N(\xi) = \Im\mathfrak{l}_N(j\xi)$ we could also consider the function $\Im\mathfrak{l}_N$ as the mapping $\Im\mathfrak{l}_N : PV \simeq \mathbb{C}P^{n-1} \to 2\mathbb{Z}_+$. The function $\Im\mathfrak{l}_N(\xi)$ is upper semicontinuous by ξ: $\overline{\lim\limits_{\xi\to\xi_0}} \Im\mathfrak{l}_N(\xi) \le \Im\mathfrak{l}_N(\xi_0)$, $\forall \xi_0 \in V$; it's also upper semicontinuous by N: $\overline{\lim\limits_{N'\to N}} \Im\mathfrak{l}_{N'}(\xi) \le \Im\mathfrak{l}_N(\xi)$, $\forall N \in \mathcal{N}$.

Definition 4. Let us say that a linear Nijenhuis tensor $N \in \mathcal{N}_j$ is of general position if for almost all ξ we have $\Im\mathfrak{l}_N(\xi) = 2$, i.e. restrictions of N onto two-dimensional planes vanish only on complex lines save the set of two-dimensional planes of zero measure.

It is easy to show that the condition in definition 4 is really a condition of general position: the set of tensors $N \in \mathcal{N}_j$ with $\Im\mathfrak{l}_N(\xi) = 2$ for a.e. ξ is open due to semicontinuity and everywhere density is easily proved by using example 3 below (this is proved in details in appendix).

Example 3. Let us consider a bilinear skew-symmetric map $A : \wedge^2(\mathbb{R}^n) \to \mathbb{R}^n$, defined by the formula:

$$A(\xi, \eta) = \sum_{i=1}^{n} (\xi^i \eta^{i+1} - \xi^{i+1}\eta^i) e_i, \; \xi = \sum_{i=1}^{n} \xi^i e_i, \; \eta = \sum_{i=1}^{n} \eta^i e_i, \; \xi^{n+1} = \xi^1, \; \eta^{n+1} = \eta^1.$$

The tensor A satisfies the property that for a.e. vector ξ it follows from the condition $A(\xi, \eta) = 0$ that vectors ξ and η are parallel.

Define a complex structure j on $\mathbb{R}^{2n} = \mathbb{R}^n_1 \oplus \mathbb{R}^n_2$ by the formula $j(\xi \oplus \eta) = (-\eta, \xi)$, and define the corresponding antiinvariant tensor $N \in \mathcal{N}(\mathbb{R}^{2n})$, $N : \wedge^2(\mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ by the formula:

$$N(\xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2) = [A(\xi_1, \xi_2) - A(\eta_1, \eta_2)] \oplus [-A(\xi_1, \eta_2) - A(\eta_1, \xi_2)].$$

It's easy to see that $\Im_N(\xi) = 2$ for a.e. ξ (see appendix for details). $\square$

It follows from definition 4 that complex lines might be singled out from two-dimensional planes by means of a Nijenhuis tensor of general position N: $\mathbf{C}\xi = \operatorname{Ker} N(\xi, \cdot)$ for almost all $\xi \neq 0$, and for others use continuity. Hence one might define complex independency of vectors using a Nijenhuis tensor N of general position: vectors $\xi_1, \dots, \xi_r$ are complex independent if for any vector ξ from the linear span $\Lambda = <\xi_1, \dots, \xi_r>$ and any number s the equality $N(\xi_s, \xi) = 0$ implies $\xi = \text{const} \cdot \xi_s$, or the same for a small perturbation of Λ. Moreover, from any complex independent collection of $n = \dim_{\mathbf{C}} V$ vectors in V one can canonically construct by means of the tensor N a decomposition on complex lines $V = \oplus_{i=1}^n \mathbf{C}$, associating to each vector its (two-dimensional) N-annihilator (or doing the same after a small perturbation and then taking the limit when the perturbation vanishes).

Proposition 2 *Let $N \in \wedge^2 V^* \otimes V$ be a skew-symmetric tensor, antilinear regardness two almost complex structures j_1 and j_2 on a complex linear space V, $N \in \mathcal{N}_{j_1} \cap \mathcal{N}_{j_2}$. If N is a tensor of general position then $j_1 = \pm j_2$.*

Proof. Let us consider arbitrary complex line $\mathbf{C} \subset V$, i.e. two-dimensional complex space, the restriction of the Nijenhuis tensor N on which vanishes, if we suppose that it is complex generated by the vector ξ such that $\Im_N(\xi) = 2$, this is the case of general position due to definition 4; the invariance of this space regarding to j_1 and j_2 follows automatically. There exists a vector $\xi \neq 0$ on this line such that the vectors $j_1\xi$ and $j_2\xi$ are parallel. Actually, up to dilations, the complex multiplication j_k acts on a complex line by rotations:

$$\varphi \mapsto \varphi + \delta_k(\varphi) \mapsto \varphi + \pi, \quad \varphi \in S^1 = \mathbb{R}^1 (\text{mod } 2\pi).$$

Changing j_2 by $(-j_2)$, if needed, one may suppose that $\delta_1(\varphi), \delta_2(\varphi) \in (0, \pi)$. Neither of inequalities $\delta_1(\varphi) < \delta_2(\varphi)$, $\delta_1(\varphi) > \delta_2(\varphi)$ could be true for all φ since $j_1^2 = j_2^2 = -\mathbf{1}$. Therefore, there exists φ such that $\delta_1(\varphi) = \delta_2(\varphi)$, i.e. there exists a vector $\xi \neq 0$ for which $j_1\xi = \alpha j_2 \xi$, $\alpha \in \mathbb{R} \setminus \{0\}$.

For any vector $\eta \notin \mathbf{C} = \mathbf{C}\xi$ we have:

$$N(\xi, j_1\eta) = N(j_1\xi, \eta) = N(\alpha j_2\xi, \eta) = N(\xi, \alpha j_2\eta),$$

i.e. $N(\xi, j_1\eta - \alpha j_2\eta) = 0$. But $j_1\eta \in \mathbf{C}\eta$ and $j_2\eta \in \mathbf{C}\eta$ (see the remarks before the proposition that $\mathbf{C}\eta$ depends not on the structures j_1 and j_2), and also $\mathbf{C}\xi \cap \mathbf{C}\eta = \{0\}$. So due to the condition that $\Im_N(\xi) = 2$ almost everywhere we have $j_1\eta = \alpha j_2\eta$ for any vector $\eta \notin \mathbf{C}\xi$, and hence, by continuity, also for any $\eta \in V$. Thus, $j_1 = \alpha j_2$, from where $\alpha = \pm 1$. $\square$

Let us consider a linear space V, two complex structures j_1 and j_2 on it and arbitrary Nijenhuis tensor $N \in \mathcal{N}_{j_1} \cap \mathcal{N}_{j_2}$, not necessary of general position. Suppose that $N \not\equiv 0$. Then there exists a vector $\theta = N(\xi, \eta) \neq 0$. Note that $j_1 j_2\theta = N(j_1\xi, j_2\eta) = j_2 j_1\theta$. Therefore the subspace $\Theta = < \theta, j_1\theta, j_2\theta, j_1 j_2\theta >$ is j_1- and j_2-invariant. Note that Θ is decomposed into the sum of invariant subspaces:

$$\Theta = < \theta^+, j_1\theta^+ > \oplus < \theta^-, j_1\theta^- >,$$

where $\theta^+ = j_1\theta + j_2\theta$ and $\theta^- = j_1\theta - j_2\theta$ satisfy the properties $j_1\theta^+ = j_2\theta^+$, $j_1\theta^- = -j_2\theta^-$. Hence, if the subspace Θ is four-dimensional then there exist nonzero vectors $\theta^\pm$ such that $j_1\theta^\pm = \pm j_2\theta^\pm$; if on the other side Θ is two-dimensional then there exists only one of such vectors. Let us denote

$$\Pi^+ = \{\xi \,|\, j_1\xi = j_2\xi\},\ \Pi^- = \{\xi \,|\, j_1\xi = -j_2\xi\},\ \Pi = \Pi^+ \oplus \Pi^-.$$

We showed that any vector $\theta \in \operatorname{Im} N(\cdot,\cdot)$ is represented as a sum of a vector from Π^+ and a vector from Π^-. Note that any vector $\theta \in V$ is represented as a sum $\theta = \theta_{(+)} + \theta_{(-)}$, where $\theta_{(\pm)} = \frac{1}{2}(\theta \mp j_1 j_2\theta)$, where $N(\theta_{(+)}, \Pi^-) = 0$, $N(\theta_{(-)}, \Pi^+) = 0$. Let us denote

$$K^+ = < \xi \,|\, N(\xi, \Pi^-) = 0 >,\ K^- = < \xi \,|\, N(\xi, \Pi^+) = 0 > .$$

We have proved:

Proposition 3 *For the subspace $\hat{\Pi} \subset V$, generated by $\operatorname{Im} N(\cdot,\cdot)$, the following inclusion holds: $\hat{\Pi} \subset \Pi$. Moreover $\Pi^+ \subset K^+$, $\Pi^- \subset K^-$, $V = K^+ + K^-$. The intersection $K^+ \cap K^-$ coincides with the space of vectors $K = \operatorname{Ker} N(\cdot, \Pi) := \{\xi \,|\, N(\xi, \Pi) = 0\}$.* $\square$

Example 4. Let us consider a linear space $V^8 = \mathbf{C}_1 \oplus \mathbf{C}_2 \oplus \mathbf{C}_3 \oplus \mathbf{C}_4$, $\mathbf{C}_s = < \partial_{2s-1}, \partial_{2s} >$, with the standard complex structure j_1 and with a structure j_2, which differs from j_1 on basis vectors only on $\mathbf{C}_3$, where $j_2\partial_5 = \partial_6 + \partial_1$, $j_2\partial_6 = -\partial_5 - \partial_2$. Let's consider the tensor $N \in \mathcal{N}_{j_1} \cap \mathcal{N}_{j_2}$, given by the conditions:

$$N(\partial_1, \partial_3) = \partial_1,\ N(\partial_5, \partial_7) = \partial_1,\ N(\mathbf{C}_1 \oplus \mathbf{C}_2, \mathbf{C}_3 \oplus \mathbf{C}_4) = 0.$$

We have: $\Pi^- = 0$, $\Pi = \Pi^+ = \mathbf{C}_1 \oplus \mathbf{C}_2 \oplus \mathbf{C}_4$, $K^- = \operatorname{Ker} N(\cdot, \Pi) = \mathbf{C}_4$, $K^+ = V^8$, $\operatorname{Ker} N(\cdot, \cdot) = 0$. $\square$

Chapter 5

Almost complex structures of general position and their classification

In this chapter we study the general properties of Nijenhuis tensors and simplify the corollary from theorem 3 (chapter 3) for almost complex structures of general position.

Definition 5. We say that an almost complex structure j on $\mathcal{O}(x)$ is of *general position* at a point x, if for the jet of the Nijenhuis tensor the following separability property holds: for almost every vector $\xi \in T_x$ there exists a number $k = k(\xi) \geq 0$ such that with some trivial connection ∇ it follows from $d^k_{\nabla} N_j(\xi, \eta, \vec{\nu}^{(k)}) = 0$, $\eta \in T_x$, for all $\vec{\nu}^{(k)} \in S^k T_x$, that $\eta \in \mathbf{C}\xi =< \xi, j\xi >$. Moreover we suppose that the pair $(\mathcal{O}(x), j)$ is not isomorphic to the pair $(\mathcal{O}(x), -j)$.

Definition 4 for linear Nijenhuis tensor is obtained from definition 5 when $k \equiv 0$. It's easy to see that the set of almost complex structures j on $\mathcal{O}(x)$ of general position contains an open and everywhere dense set in the set of all almost complex structures; hence it follows that the set of the points, where an almost complex structure is of general position, is open. Note that similar to the linear case for an almost complex structure j of general position from any complex independent collection of n vectors in T_x one can canonically construct by means of the tensor N_j a complex polarization: $T_x = \oplus_{i=1}^{n} \mathbf{C}$.

Remark 4. An almost complex structure j of general position on $\mathcal{O}(x)$ is not isomorphic to the structure $(-j)$. Actually, for the invariant, introduced in chapter 2, we have: $N_{-j}^{(2)} = -N_j^{(2)}$, and for a general almost complex structure j tensors $N_j^{(2)}$ and $(-N_j^{(2)})$ (on the fixed tangent space) are nonisomorphic. Note that $(T_x, j_x) \simeq (T_x, -j_x)$, and $N_{-j} = N_j$, from where it follows that $j \simeq (-j) \,(\mathrm{mod}\ \mu_x^2)$ on $\mathcal{O}(x)$, but almost always $j \not\simeq (-j) \,(\mathrm{mod}\ \mu_x^3)$ and the tensor $N_j^{(2)}$ is the first generating invariant in $\mathcal{A}_j^\infty$, which distinguish, in general case, almost complex structures j and $(-j)$. $\square$

Example 5. Consider the almost complex structure in $\mathbb{R}^4$, defined by the equalities:

$$\begin{aligned} j\partial_1 &= \partial_2, & j\partial_3 &= \partial_4 + (x^2)^2\partial_1 + \varepsilon(x^3)^2\partial_2, \\ j\partial_2 &= -\partial_1, & j\partial_4 &= -\partial_3 - (x^2)^2\partial_2 + \varepsilon(x^3)^2\partial_1. \end{aligned}$$

For $\varepsilon = 0$ almost complex structures j and $(-j)$ are isomorphic, and for $\varepsilon \neq 0$ are not. It's easy to see that the given almost complex structure for any $\varepsilon \neq 0$ is of general position. $\square$

Theorem 6. *Let us suppose that almost complex structures j_L on $\mathcal{O}_L(x)$ and j_M on $\mathcal{O}_M(y)$ are of general position.*

1°. *Let us suppose that there exists a mapping $u : \mathcal{O}_L(x) \to \mathcal{O}_M(y)$, which conjugates N_{j_L} with N_{j_M} and is a diffeomorphism on the image. Then u is either pseudoholomorphic or antipseudoholomorphic mapping: $j_M \circ u_* = \pm u_* \circ j_L$.*

2°. *If the Nijenhuis tensors are conjugated formally, $N_{j_M} \circ u_*^{\wedge 2} \equiv u_* \circ N_{j_L}$ $(\mathrm{mod}\, \mu_{x,y}^\infty)$, and the dimensions of manifolds L and M are equal, $2l = 2m$, then the almost complex structure j_L is formally equivalent to exactly one of the structures j_M or $(-j_M)$.*

Proof. Let us consider the case of smooth conjugacy of the Nijenhuis tensors; the formal one is obtained by passing on to series. Let $N_{j_M} \circ \Lambda^2 u_* = u_* \circ N_{j_L}$. One may choose complex parallelizations $T_{x'}L = \oplus_1^l \mathbb{C}$ and $T_{y'}M = \oplus_1^m \mathbb{C}$ such that the mapping u_* includes one into the other. Therefore the image $u(L)$ is j_M-invariant. Hence one can consider the case of the manifolds of the same dimensions $2l = 2m$ and suppose that u is a diffeomorphism. Thus we have at a point $y' \in u(L) \cap M$ the Nijenhuis tensor $N_{j_M} = u_* \circ N_{j_L} \circ (u_*^{\wedge 2})^{-1}$ and two complex structures j_M and $u_* \circ j_L \circ u_*^{-1}$ on $T_{y'}$, regardness to which it is antilinear. Note that the statement of proposition 2 holds true if instead of the tensor N one considers the jet $\oplus_k d^k N_j$ of the Nijenhuis tensor of general position. Therefore, $j_M = \pm u_* j_L$, and since an almost complex structure j of general position is not isomorphic to $(-j)$, the theorem is proved. $\square$

Remark 5. The theorem is not true for almost complex structures not of general position. If j_L on $\mathcal{O}_L(x)$ and j_M on $\mathcal{O}_M(y)$ are structures of general position then the Nijenhuis tensors of almost complex structures $j_L \oplus j_M$ and $j_L \oplus (-j_M)$ on $\mathcal{O}_L(x) \times \mathcal{O}_M(y)$ coincide, while the structures themselves are neither isomorphic nor antiisomorphic. $\square$

Chapter 6

Two special cases

6.1 Almost complex structures in $\mathbb{R}^4$

Let us consider an almost complex structure j in $\mathbb{R}^4$ in a neighborhood of zero. Suppose that $(N_j)_0 \neq 0$. In this case the image $\operatorname{Im} N_j(\cdot,\cdot)$ is two-dimensional and complex generated by a vector $N_j(\xi,\eta) \neq 0$ for any two complex independent vectors ξ and η. Consider a two-dimensional distribution $\Pi^2_x = \operatorname{Im}(N_j)_x \subset T_x = T_x\mathbb{R}^4$. This distribution is an invariant of almost complex structure and hence invariants of this distribution lead to invariants of the almost complex structure.

An important invariant of a distribution is the Tanaka invariant ([T], [Y]): a collection of graded Lie algebras $\mathfrak{g}(x)$ for each point x, associated to the filtered algebras $\{D^p(x) \subset T_x\}_{p\geq 1}$, where the module of sections $\mathcal{D}^{(p)}$ of the distribution D^p is defined as $(p-1)$-th derivative: $\mathcal{D}^{(p)} = \partial^{(p-1)}\mathcal{D}^{(1)} = \partial^{(p-2)}\mathcal{D}^{(1)} + [\mathcal{D}^{(1)}, \partial^{(p-2)}\mathcal{D}^{(1)}]$, $[\cdot,\cdot]$ being the commutator of the vector fields, and $\partial^{(0)}\mathcal{D}^{(1)} = \mathcal{D}^{(1)}$ is the sections module of the initial distribution D^1; the Lie product is induced by the commutator of vector fields.

Let us suppose that the first derivative of the distribution Π^2_* is nontrivial at the point: $(\partial^{(1)}\Pi^2)_0 \neq \Pi^2_0$. Then we have a three-dimensional distribution $\Pi^3 = \partial^{(1)}\Pi^2$ in $\mathcal{O}(0)$. There exist vectors $\xi_i = \xi_i(x) \neq 0$, $i = 1,2,3$, in the space $\Pi^3_x \subset T_x$ such that $N_j(\xi_1,\xi_3) = \xi_1$, $N_j(\xi_2,\xi_3) = -\xi_2$. Actually, for any $\xi_3 \in \Pi^3 \setminus \Pi^2$ the mapping $\eta \mapsto N(\eta,\xi_3)$ is an orientation reversing isomorphism of Π^2, and hence there exist two one-dimensional invariant subspaces. Consider a vector ξ_1 on one of them. We have: $N(\xi_1,\xi_3) = f\xi_1$, $f \neq 0$. Let's change the vector $\xi_3 \mapsto \frac{1}{f}\xi_3$. Then $N(\xi_1, \xi_3) = \xi_1$ and $N(j\xi_1, \xi_3) = -j\xi_1$, i.e. we may take $\xi_2 = j\xi_1$.

Denote by $\Upsilon_1 =< \xi_1 >$ and $\Upsilon_2 =< \xi_2 >$ the one-dimensional subspaces generated by the vectors ξ_1 and ξ_2, $\Pi^2_x = (\Upsilon_1)_x \oplus (\Upsilon_2)_x$. Let us note that if the half-space $(\Pi^3_x)^+ \subset \Pi^3_x \setminus \Pi^2_x$, in which lies the vector ξ_3, is fixed then the vector is defined up to Π^2_x-shifts: $\xi_3 \mapsto \xi_3 + \alpha_1\xi_1 + \alpha_2\xi_2$. If one changes the half-space: $\xi_3 \mapsto \tilde{\xi}_3 = -\xi_3$, then the vectors ξ_1 and ξ_2 (and hence the distributions Υ_1 and Υ_2) interchanges: $\tilde{\xi}_1 = \pm\xi_2$, $\tilde{\xi}_2 = \pm\xi_1$. Selecting a half-space in $\Pi^3_x \setminus \Pi^2_x$, i.e. an orientation on $\Theta^1_x = \Pi^3_x/\Pi^2_x$, we fix which one of the distributions Υ_1 and Υ_2 is the first and which is the second; changing this orientation we change the numeration. In other words, there is a canonical orientation on the two-dimensional space $\Theta^1_x \times P(\Pi^2_x)$. Let us call this orientation the Θ-orientation. Note also that there's singled out a pair of two-dimensional affine subspaces in the space Π^3_x: $\{\pm\xi_3 + \alpha_1\xi_1 + \alpha_2\xi_2\}$, i.e. there's fixed a metric on Θ^1_x. Let us call it the Θ-metric.

Consider now the cofactor $\Xi^1_x = T^4_x/\Pi^3_x$ of the subspace $\Pi^3_x \subset T_x$. There exists a vector $\xi_4 \notin \Pi^3_x$ such that $N_j(\xi_1, \xi_4) = \xi_2$, $N_j(\xi_2, \xi_4) = -\xi_1$. In its half-space $T^4_x \setminus \Pi^3_x$ it is determined up to Π^2_x-shifts: $\xi_4 \mapsto \xi_4 + \alpha_1\xi_1 + \alpha_2\xi_2$, and so there exists a natural metric on Ξ^1_x. Let us call it the Ξ-metric. Under the change of half-space $\xi_4 \mapsto \tilde{\xi}_4 = -\xi_4$ in one of two subspaces Υ_1 or Υ_2 the orientation changes: $\xi_1 \mapsto -\xi_1$ или $\xi_2 \mapsto -\xi_2$. Therefore there's determined the natural orientation of the space $\Upsilon^1_1 \times \Upsilon^1_2 \times \Xi^1$. Let us call it the Ξ-orientation.

Let us call the $\Upsilon\Theta\Xi$-invariant the pair of distribution Υ_i and Θ- and Ξ-metrics and orientations. From the arguments above together with theorem 6 it follows

Theorem 7. *Let the Nijenhuis tensors of almost complex structures j_1 in $\mathcal{O}_{\mathbf{R}^4}(x)$ and j_2 in $\mathcal{O}_{\mathbf{R}^4}(y)$ do not equal to zero and the first derivative of the distribution Π^2 is nontrivial, $\partial^{(1)}\Pi^2 \neq \Pi^2$. Pseudoholomorphic or antipseudoholomorphic mapping $u : (\mathcal{O}_{\mathbf{R}^4}(x), j_1) \to (\mathcal{O}_{\mathbf{R}^4}(y), j_2)$ exists if and only if there exists a mapping $\mathcal{O}_{\mathbf{R}^4}(x) \to \mathcal{O}_{\mathbf{R}^4}(y)$, which transforms one $\Upsilon\Theta\Xi$-invariant to the other.* □

Remark 6. In the case if the second derivative of the distribution Π^2 does not coincide with the first, $\partial^{(2)}\Pi^2 \neq \partial^{(1)}\Pi^2 \neq \Pi^2$, i.e. $(\partial^{(2)}\Pi^2)_x = T^4_x$, the

Tanaka invariant ([T]) — graded Lie algebra — has the underlying space of the form

$$\varrho(x) = \varrho(x)_1 \oplus \varrho(x)_2 \oplus \varrho(x)_3 \simeq \Pi_x^2 \oplus \Theta_x^1 \oplus \Xi_x^1.$$

In this case, because of the gradation, the Lie product is determined by a 2-form on Π_x^2 with values in Θ_x^1 and by 1-form on Π_x^2 with values in $\mathrm{Hom}(\Theta_x^1, \Xi_x^1)$. Due to existence of the Θ- and Ξ-metrics and orientations and due to the decomposition $\Pi_x^2 = (\Upsilon_1)_x \oplus (\Upsilon_2)_x$, the Lie algebra structure on $\varrho(x)$ is given by elements $\omega_x^2 \in (\Upsilon_1)_x^* \otimes (\Upsilon_2)_x^* = \mathrm{Hom}(\Upsilon_1, \Upsilon_2^*)_x$ and $\omega_x^1 \in (\Upsilon_1)_x^* \oplus (\Upsilon_2)_x^*$. Thus the Tanaka invariant gives us two additional invariants: ω^1 and ω^2. $\square$

6.2 Lie almost complex structures

Let us consider an almost complex structure j and its Nijenhuis tensor N_j. This tensor defines a bilinear skew-symmetric map $N_j : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$ on the module $\mathcal{D}$ of vector fields. In general this map is not a Lie product. Actually, from the Jacobi identity for the elements ξ, η, $j\zeta$

$$N(\xi, N(\eta, j\zeta)) + N(\eta, N(j\zeta, \xi)) + N(j\zeta, N(\xi, \eta)) = 0$$

it follows that

$$jN(\xi, N(\eta, \zeta)) + jN(\eta, N(\zeta, \xi)) - jN(\zeta, N(\xi, \eta)) = 0,$$

which together with the Jacobi identity for ξ, η, ζ gives:

$$N(\xi, N(\eta, \zeta)) \equiv 0. \tag{25}$$

Thus $(\mathcal{D}, N(\cdot,\cdot))$ is a Lie algebra iff identity (25) holds. Let us denote by $\Pi_x \subset T_x$ the subspace generated by $\mathrm{Im}\, N(\cdot,\cdot)$ and denote $A_x = \{\xi \mid N(\xi, \cdot) \equiv 0\}$. Formula (25) is equivalent to the inclusion of (j-invariant) subbundles $\Pi \subset A \subset TX$.

Definition 6. Let us call an almost complex structure j on X Lie almost complex structure if its Nijenhuis tensor N_j defines a Lie algebra structure on the module $\mathcal{D}$.

From theorem 6 (chapter 5) and the arguments above it follows

Theorem 8. *A Lie almost complex structure j defines a solvable Lie algebra $\mathcal{G} = (\mathcal{D}, N_j(\cdot,\cdot))$ with the solvability rank 2: $\mathcal{G}^{(2)} = 0$. Lie almost complex structures of general position are equivalent or antiequivalent (formally or smooth) iff the corresponding Lie algebras $(\mathcal{D}, N(\cdot,\cdot))$ are equivalent.* □

Thus the theorem reduces the classification of Lie almost complex structures to the classification of Lie algebras bundle (Lie algebras structures on the tangent spaces $T_{x'}X$), i.e. to the analysis of Lie multiplication deformations; let us note that not all such deformations realize, see remark 3.

Remark 7. Let us consider *Nijenhuis (sub)algebra* $\mathcal{A}_j^{\mathcal{N}}$, which is generated by the complex structure j by means of the algebraic commutator $[A,B] = B\bar{\wedge}A - (-1)^{\deg A \deg B} A\bar{\wedge}B$ and the Nijenhuis bracket $[\![\cdot,\cdot]\!]$, in the algebra of vector-valued forms $\Omega^\bullet \otimes \mathcal{D}$ on the manifold X (see [FN], [ALV], the notations are similar to [KS]). Here $\Omega^\bullet$ is the algebra of scalar valued differential forms and $\mathcal{D}$ is the module of vector fields. Note that in the case of Lie almost complex structure $\mathcal{A}_j^{\mathcal{N}} = < j, N_j, j \circ N_j >$. Actually, as formulae (2.9), (2.10), (2.16), (5.6), (5.8), (5.22), (5.23) from [FN] together with the Jacobi identity $N_j\bar{\wedge}N_j = 0$ for N_j show, the following equalities hold true:

$$
\begin{aligned}
&[j,j] = 0,\ [j,N_j] = -3j\bar{\wedge}N_j = \frac{3}{2}N_j\bar{\wedge}j = -3j\circ N_j,\ [j, j\circ N_j] = 3N_j,\\
&[N_j,N_j] = 2N_j\bar{\wedge}N_j = 0,\ [N_j, j\circ N_j] = 0,\ [j\circ N_j, j\circ N_j] = 0,\\
&[\![j,j]\!] = N_j,\ [\![j,N_j]\!] = 0,\ [\![j, j\circ N_j]\!] = -\frac{1}{2}N_j\bar{\wedge}N_j = 0,\\
&[\![N_j,N_j]\!] = 0,\ [\![N_j, j\circ N_j]\!] = 0,\ [\![j\circ N_j, j\circ N_j]\!] = 0.
\end{aligned}
$$

It follows from the example below that the values set of $(\mathcal{A}_j^{\mathcal{N}})_x$ at the point x is not sufficient for formal classification of the almost complex structure, i.e. $\mathcal{A}_j^{\mathcal{N}} \neq \mathcal{A}_j^{\infty}$, see the beginning of chapter 2. □

Example 6. Let us consider the almost complex structure defined by the formulae:

$$
\begin{aligned}
&j\partial_1 = \partial_2, && j\partial_3 = \partial_4 + f(x^5)\partial_1, && j\partial_5 = \partial_6,\\
&j\partial_2 = -\partial_1, && j\partial_4 = -\partial_3 - f(x^5)\partial_2, && j\partial_6 = -\partial_5.
\end{aligned}
$$

The Nijenhuis tensor on the basis vectors is given by the identities:

$$N_j(\partial_1,\cdot) = N_j(\partial_2,\cdot) = N_j(\partial_3,\partial_4) = N_j(\partial_5,\partial_6) = 0,$$

$$N_j(\partial_3, \partial_5) = -N_j(\partial_4, \partial_6) = f'(x^5)\partial_2,\ N_j(\partial_3, \partial_6) = N_j(\partial_4, \partial_5) = f'(x^5)\partial_1.$$

One can see that the structure j is not formally classified by the algebra $(\mathcal{A}_j^{\mathcal{N}})_x$. Actually, for $f(x^5) = \varepsilon \cdot x^5$, there is an additional independent invariant $(dN_j)_0$, and for $f(x^5) = x^5 + \varepsilon(x^5)^2$ the algebra $\mathcal{A}_j^{\mathcal{N}}$ also is not sufficient for distinguishing the structures corresponding to different ε, even though the Nijenhuis tensors are conjugated in all the point from the neighborhood of x. $\square$

Remark 8. Let us consider the Nijenhuis tensor of a Lie almost complex structure j in $\mathcal{O}(x)$. There exists an increasing j-invariant filtration $\Theta_k \subset T_x$, where Θ_k is generated by $\operatorname{Im} d^k N(\cdot, \cdot, *)$. One can relate to a point x the following invariant: the associated graded (j-)complex Lie algebra regardness N-product — $\oplus V_k$, $V_k = \Theta_k/\Theta_{k-1}$. $\square$

Appendix A

Linear Nijenhuis tensors of general position

Here we prove that tensors $N \in \mathcal{N}_j$ satisfying the condition of definition 4 of chapter 4 are actually the tensors of general position in $\mathcal{N}_j$. Let $V \simeq \mathbb{C}^n$ be the complex linear space.

Theorem 9. *The set $\mathcal{U}_j$ of all tensors $N \in \mathcal{N}_j(V)$ such that $\Im_N(\xi) = 2$ for almost all $\xi \in V$ is open and everywhere dense in $\mathcal{N}_j$.*

Proof. $1°$. Let us fix a complex basis on V, i.e. we fix an isomorphism $V = \mathbb{C}^n$. Let us also fix a Euclidean structure on V in which the basis is orthonormal; in particular the automorphism j is an orthogonal transformation. Consider an $(n-1)$-dimensional complex subspace $\Pi \subset V$, i.e. $(2n-2)$-dimensional subspace invariant under j. Consider the mapping $N_\xi : \Pi \to V$, $N_\xi(\eta) = N(\xi, \eta)$, where $\xi \notin \Pi$. The last condition means that the complex line $\mathbb{C}\xi =< \xi, j\xi >$ is transversal to Π. Let us fix an orthonormal complex basis $\eta_1, \ldots, \eta_{2n-2}$ in Π, $\eta_{2k} = j\eta_{2k-1}$. Define the function $\alpha_N : V \setminus \Pi \to \mathbb{R}$ as the $(2n-2)$-dimensional volume of the body $\{\sum\limits_{k=1}^{2n-2} x_k \cdot N_\xi(\eta_k) \mid 0 \leq x_k \leq 1\}$. The condition $\Im_N(\xi) = 2$ of definition 4 is equivalent to $\alpha_N(\xi) > 0$ for $\xi \in V \setminus \Pi$.

Note that the square of the function $\alpha_N(\xi)$ is a rational fracture of the variable $\xi = (\xi_1, \ldots, \xi_{2n})$. So if $\alpha_N \not\equiv 0$ then the set of ξ such that $\alpha_N(\xi) = 0$ is a hypersurface (actually a cone over a surface) and so has the measure 0.

2°. Show that the set $\mathcal{U}_j$ in $\mathcal{N}_j$ is nonempty. Really, consider the tensor $N \in \mathcal{N}_j$ defined by

$$N(\vec{z}, \vec{w}) = \left(0, \begin{vmatrix} \bar{z}_1 & \bar{z}_2 \\ \bar{w}_1 & \bar{w}_2 \end{vmatrix}, \begin{vmatrix} \bar{z}_1 & \bar{z}_3 \\ \bar{w}_1 & \bar{w}_3 \end{vmatrix}, \ldots, \begin{vmatrix} \bar{z}_1 & \bar{z}_n \\ \bar{w}_1 & \bar{w}_n \end{vmatrix}\right),$$

where $\vec{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\vec{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and $\bar{z}_k = x_k - jy_k$ for $z_k = x_k + jy_k$. One easily sees that if $z_1 \neq 0$ and $\vec{w} \neq \lambda \vec{z}$ for any $\lambda \in \mathbb{C}$ then $N(\vec{z}, \vec{w}) \neq 0$. So here we have $\alpha_N(\xi) \neq 0$ for all $\xi \notin \Pi = \{\vec{z} \mid z_1 = 0\}$.

3°. Show that $\mathcal{U}_j$ is open in $\mathcal{N}_j$. Let $N \in \mathcal{U}_j \subset \mathcal{N}_j$. Consider the set $\{\xi \mid \alpha_N(\xi) = 0,\ |\xi| = 1\}$. This is a hypersurface in $S^{2n-1} \subset V$. Let $\mathcal{W}$ be a small neighborhood of it. Let $a > 0$ be the minimum of α_N on the closure $\overline{S^{2n-1} \setminus \mathcal{W}}$. There exists a neighborhood $\mathcal{Y}$ of N in $\mathcal{N}_j$ such that for any $N' \in \mathcal{Y}$ we have

$$\alpha_{N'}(\xi) > \frac{a}{2},\ \xi \in \overline{S^{2n-1} \setminus \mathcal{W}}.$$

Hence $\alpha_{N'} \not\equiv 0$ and $N' \in \mathcal{U}_j$ according to 1°, and we have proved the desired inclusion $\mathcal{Y} \subset \mathcal{U}_j$.

4°. Let us prove that $\mathcal{U}_j$ is everywhere dense in $\mathcal{N}_j$. Let $N \in \mathcal{N}_j$ and let $N_0 \in \mathcal{U}_j$ be the tensor constructed in 2°. Consider the 1-parameter family $N_\varepsilon = \varepsilon N + (1-\varepsilon) N_0 \in \mathcal{N}_j$. We have: $N_1 = N$. The square of the function $\alpha_{N_\varepsilon}(\xi)$ is a rational fracture of ε. This function is not equal to zero for $\varepsilon = 0$ for almost all ξ. So it could not be zero identically. In particular there exists a number $\varepsilon_0 > 0$ such that for any $\varepsilon \in (1 - \varepsilon_0, 1)$ the function α_{N_ε} does not vanish identically, i.e. $N_\varepsilon \in \mathcal{U}_j$. $\square$

In the same way one could prove the theorem "without j". Let the set $\mathcal{V}$ consist of all vector two-forms $A \in \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ such that for almost all $\xi \in \mathbb{R}^n$ the equality $A(\xi, \eta) = 0$ implies that $\eta = \lambda \xi$ for some $\lambda \in \mathbb{R}$.

Theorem 10. *$\mathcal{V}$ is open and everywhere dense in $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.* $\square$

One of the forms $A \in \mathcal{V}$ is given in the example 3 from chapter 4. We prolong this tensor A to the tensor N on the complexified space $\mathbb{R}^n_{\mathbb{C}} = \mathbb{C}^n \simeq \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^n \oplus j\mathbb{R}^n$ as is shown in the example. Let us show that $\Im_N = 2$ almost everywhere. Really, consider $\xi_1 = (1, \ldots, 1)$, $\eta_1 = (1, \ldots, 1)$, $\Pi = \{(0, *, \ldots, *) \oplus (0, *, \ldots, *)\} \subset \mathbb{R}^n \oplus j\mathbb{R}^n$. Let $\xi_2 = (0, \xi_2^2, \ldots, \xi_2^n)$ and $\eta_2 = (0, \eta_2^2, \ldots, \eta_2^n)$ be two vectors from Π ($\xi_2^1 = \eta_2^1 = 0$). We have:

$$
\begin{aligned}
N(\xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2) = & \left[\sum_{k=1}^{n} (\xi_2^{k+1} - \xi_2^k) e_k - \sum_{k=1}^{n} (\eta_2^{k+1} - \eta_2^k) e_k \right] \\
& \oplus \left[- \sum_{k=1}^{n} (\eta_2^{k+1} - \eta_2^k) e_k - \sum_{k=1}^{n} (\xi_2^{k+1} - \xi_2^k) e_k \right].
\end{aligned}
$$

So if $N_{\xi_1 \oplus \eta_1}(\xi_2 \oplus \eta_2) = 0$ then $\sum\limits_{k=1}^{n} (\xi_2^{k+1} - \xi_2^k) e_k = 0$, $\sum\limits_{k=1}^{n} (\eta_2^{k+1} - \eta_2^k) e_k = 0$, from where we have $\xi_2^k = \eta_2^k = 0$, $k = 2, \ldots, n$. Hence $\alpha_N(\xi_1 \oplus \eta_1) > 0$ and $\Im_N(\xi_1 \oplus \eta_1) = 2$, which means that N is of general position (see the proof of theorem 9).

Remark 9. We can construct tensors $A \in \mathcal{V}$, $A : \wedge^2 \mathbb{R}^n \to \mathbb{R}^n$, such that $A(\xi, \eta) = 0$ only for parallel vectors ξ and η. Let us denote them by $\mathcal{V}_0$. For $n = 2$: $\mathcal{V}_0 = [\wedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n] \setminus \{0\}$. For $n = 3$ we can take as example of A the usual vector product $\mathbb{R}^3 \wedge \mathbb{R}^3 \to \mathbb{R}^3$. Let us consider the case $n = 4$. We have the Plücker-Grassmann mapping:

$$
PG : \wedge^2 \mathbb{R}^4 \to \mathbb{R}^4, \ (\xi . \eta) \mapsto \theta = (\theta_{rs})_{1 \le r < s \le 4},
$$

where $\theta_{rs} = \begin{vmatrix} \xi^r & \xi^s \\ \eta^r & \eta^s \end{vmatrix}$. The image $\mathrm{Im}(PG)$ is the quadric (cone) given by the equation

$$
\mathcal{C} : \ \{\theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23} = 0\}.
$$

Let us take the four-dimensional subspace $\{\theta_{12} = -\theta_{34}, \ \theta_{13} = \theta_{24}\}$ in $\mathbb{R}^6$. The orthogonal projection pr on this subspace is given by the formulae:

$$
\begin{aligned}
\mu_1 &= \theta_{12} - \theta_{34}, \\
\mu_2 &= \theta_{13} + \theta_{24}, \\
\mu_3 &= \theta_{14}, \\
\mu_4 &= \theta_{23}.
\end{aligned}
$$

So by composition $\mathrm{pr} \circ PG$ we obtain the map $A : \wedge^2 \mathbb{R}^4 \to \mathbb{R}^4$. This map satisfies the property that $A(\xi, \eta) = 0$ only for parallel ξ and η. Actually, in this case $\theta_{12} = \theta_{34}$, $\theta_{13} = -\theta_{24}$, $\theta_{14} = 0$, $\theta_{23} = 0$. Substituting these equalities into the formula for $\mathcal{C}$ we have

$$
\theta_{12}^2 + \theta_{13}^2 = 0,
$$

i.e. $\theta_{rs} = 0$ for all $1 \leq r < s \leq 4$, which means that ξ is parallel to η.

As in the formula of example 3 we could construct a mapping $N_A : \wedge^2 \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^n \oplus j\mathbb{R}^n$, $N_A \in \mathcal{N}_j$. But this map does not satisfy the property that $N_A(\xi, \eta) = 0$ only for $\mathbb{C}$-dependent vectors ξ and η (note that in difference with definition 4 here we require that the property holds not almost always but always). Actually, the author do not know whether there is for any n the tensor $N \in \mathcal{N}_j^{(n)}$ such that $N(\xi, \eta) = 0$ implies $\xi \in \mathbb{C}\eta$ unless $\eta = 0$. $\square$

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