Pseudoholomorphic mappings and Kobayashi hyperbolicity

Boris S. Kruglikov and Marius Overholt

Abstract

We extend the definition of the Kobayashi pseudodistance to almost complex manifolds and show that its familiar properties are for the most part preserved. We also study the automorphism group of an almost complex manifold. We give special consideration to almost complex structures tamed by some symplectic forms. The notions and pseudoholomorphic curves involved are illustrated in some examples.

Introduction

The Poincaré metric on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is a Riemannian metric

$$|v| = \frac{|v|_{euc}}{1-|z|^2}$$

conformal with the Euclidean metric $|\cdot|_{euc}$, that induces a distance d on $\mathbb{D}$ with the remarkable property that every holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$ is distance nonincreasing in d. This fact, discovered in 1915 by Pick [Pi] is an invariant formulation of the Schwarz lemma. By means of holomorphic mappings of the unit disk into a complex manifold M, Kobayashi in 1967 used the distance d to define a pseudodistance d_M on M. This has the property that any holomorphic mapping from a complex manifold N to a complex manifold M is distance nonincreasing with respect to d_N and d_M. When this Kobayashi pseudodistance d_M is a distance, it can be used to obtain information about holomorphic mappings to or from M; in this situation M is said to be (Kobayashi-) hyperbolic. Some references for Kobayashi hyperbolicity are [JP, Ko1, Ko2, La, NO].

An almost complex manifold (M^{2n}, J) is a smooth manifold of dimension $2n$ equipped with a smooth field J of automorphisms of the tangent spaces of M^{2n}, which satisfy $J^2 = -\mathbf{1}$. Such an assignment of automorphisms J is called an almost complex structure on M^{2n}. A pseudoholomorphic mapping of the unit disk $\mathbb{D}$ into (M^{2n}, J) is a smooth mapping $f : \mathbb{D} \to (M^{2n}, J)$ which satisfies $J \circ f_* = f_* \circ J_0$ where J_0 is the almost complex structure on $\mathbb{D}$ induced by the standard complex structure. Using such pseudoholomorphic disks, the definition of the Kobayashi pseudodistance can be carried over to almost complex manifolds. Many proofs of standard results about d_M in the almost complex case are similar to known ones from the complex situation; results of this kind will be merely noted in passing where appropriate. However some facts need a treatment involving new ideas. We will emphasise such results. To deal with them we apply the Gromov pseudoholomorphic curve technique which was considerably developed in symplectic geometry during the past decade ([Gr, MS]).

A symplectic manifold (M^{2n}, ω) is a smooth manifold M^{2n} together with a closed and nondegenerate ($\omega^n \neq 0$) 2-form ω on M^{2n}. In the famous paper [Gr] Gromov showed how to use almost complex structures to prove new results in symplectic geometry. He proved the existence, on each symplectic manifold, of almost complex structures J satisfying $\omega(v, Jv) > 0$ for any $v \neq 0$ in TM^{2n}. Such an almost complex structure is called tame, and ω is said to tame J. Pseudoholomorphic mappings $f : S \to (M^{2n}, J)$ from Riemann surfaces S yield information about (M^{2n}, ω) when ω tames J.

In the paper [Ban] Bangert used a technique of hyperbolic complex analysis called Brody reparametrization to prove that the standard symplectic torus (T^{2n}, ω_0) contains a entire J-holomorphic line for any J taming ω_0. In the preliminary version of this paper [KO] we independently treated the Brody criterion for the nonintegrable almost complex case. We consider the connection of our results with that of Bangert and outline an alternative proof of his theorem in section 4.

The taming condition has some bearing on hyperbolicity of almost complex manifolds. Roughly speaking, two conditions – tameness and hyperbolicity – are opposite to each other in the large. This makes possible connections with symplectic geometry rather awkward as we discuss in the conclusion.

This paper was initiated during the visit of the first author to the University of Tromsoe. He thanks professor V. Lychagin and other mathematicians from the Department for their hospitality and kind attention.

1. Definition of the pseudometric

Let $\mathbb{C}$ denote the complex plane with its standard complex structure, $\mathbb{D}$ the open unit disk in it and $e = 1 \in T_0\mathbb{D}$. Let (M^{2n}, J) be an almost complex manifold, which in this paper will be taken to be connected and C^∞. This means that the smooth field of automorphisms $J \in T^*M \otimes TM$ satisfies $J^2 = -\mathbf{1}$. A Kobayashi chain joining two points $p, q \in M$ is a sequence of pseudoholomorphic mappings

$$f_k : \mathbb{D} \to (M^{2n}, J), \quad k = 1, ..., m,$$

and points $z_k, w_k \in \mathbb{D}$ such that $f_1(z_1) = p$, $f_m(w_m) = q$ and $f_k(w_k) = f_{k+1}(z_{k+1})$. The *Kobayashi pseudodistance* from p to q is defined by

$$d_M(p, q) = \inf \sum_{k=1}^{m} d(z_k, w_k)$$

where the infimum is taken over all Kobayashi chains joining p to q, if there exists some Kobayashi chain joining p to q, and is defined to be $+\infty$ otherwise. It is well known that $d_{\mathbb{C}} \equiv 0$ and $d_{\mathbb{D}} \equiv d$. We record the very basic properties of d_M:

Proposition 1. (i) *The function $d_M : M \times M \to \mathbb{R}$ is nonnegative, symmetric and satisfies the triangle inequality, i.e. it is a pseudodistance.*
(ii) *Pseudoholomorphic mappings between almost complex manifolds are distance nonincreasing with respect to the Kobayashi pseudodistance.*

As in the case of complex manifolds, d_M is finite on any almost complex manifold, but this requires an existence theorem for pseudoholomorphic disks due to Nijenhuis and Woolf [NW].

Theorem 1. (i) *The Kobayashi pseudodistance d_M is finite and continuous on $M \times M$ for any almost complex manifold (M^{2n}, J).*
(ii) *If d_M is a distance, it induces the standard topology on M.*

Proof. We must first prove that any two points in M can be joined by a Kobayashi chain. It is enough to prove that for any point $p \in M$ there is a neighborhood U_p of p such that any point q in U_p can be joined to p by a single pseudoholomorphic disk. For then the set of points that can be joined by a Kobayashi chain to some fixed point is open and nonempty, and its complement is open, so every point can be joined to the fixed point since M is connected.

The problem is now local, so we consider $(\mathbb{R}^{2n}, J)$. Let $v \in T_0\mathbb{R}^{2n} = \mathbb{R}^{2n}$. By theorem III of [NW], there is some neighborhood V of 0 in $\mathbb{R}^{2n}$ such that if $v \in V$, then there exists a pseudoholomorphic mapping $f : \mathbf{D} \to (\mathbb{R}^{2n}, J)$ with $f(0) = 0$ and $f_*(0)e = v$; this mapping could be chosen canonically. We write this mapping as $f(z; v)$ to show its dependence on v. We denote its differential with respect to z by $f_*(z; v)$. By 5.4a of [NW], $f : \mathbf{D} \times V \to \mathbb{R}^{2n}$ is a C^∞ mapping. It satisfies $f(0; v) = 0$ and $f_*(0; v)e = v$. In addition, by 5.2a of [NW] , it satisfies $f(z; 0) = 0$ on the unit disk. Since

$$\frac{\partial f_*(0; v)e}{\partial v} = I$$

there is some neighborhood W of 0 in $\mathbf{D}$ such that for every $\zeta \in W$, the matrix

$$\left.\frac{\partial f_*(\zeta; v)e}{\partial v}\right|_{v=0}$$

is of full rank. Now

$$\left.\frac{\partial f(\zeta; v)}{\partial v}\right|_{v=0} = \zeta \left.\frac{\partial f_*(\zeta; v)e}{\partial v}\right|_{v=0} + O(|\zeta|^2),$$

where multiplication is with respect to the complex structure on $\mathbb{R}^{2n} \simeq \mathbf{C}^n$ induced by the linearization. So there exists some $\zeta_0 \in W \backslash \{0\}$ such that

$$\left.\frac{\partial f(\zeta_0; v)}{\partial v}\right|_{v=0}$$

is a matrix of full rank. Then $f(\zeta_0; V)$ contains an open neighborhood of 0 in $\mathbb{R}^{2n}$, because $f(\zeta_0; 0) = 0$. So if q is in this neighborhood, there is some $v \in V$ such that $f(\cdot; v)$ is a pseudoholomorphic disk joining 0 and q.

Since

$$|d_M(p, q) - d_M(p_0, q_0)| \leq d_M(p, p_0) + d_M(q, q_0)$$

by the triangle inequality, to prove continuity of d_M it is enough to prove continuity of the mappings $d(\cdot, p_0)$ for all $p_0 \in M$. The reasoning above shows that for every $\varepsilon > 0$ there exists some neighborhood U of p_0 such that for some $\zeta_0 \in \mathbb{D}$ with $d(0, \zeta_0) = \varepsilon$ the image $f(\zeta_0; V)$ contains U and thus $d_M(p, p_0) \leq \varepsilon$ for any $p \in U$. The proof that d_M induces the standard topology on M if it is a distance can now be carried through by the method of Barth [Ba]. $\square$

2. Hyperbolicity

An almost complex manifold (M^{2n}, J) is *Kobayashi hyperbolic* if the Kobayashi pseudodistance d_M is a distance. It is *Brody hyperbolic* if any pseudoholomorphic mapping $f : \mathbb{C} \to M$ is constant. It is clear that Kobayashi hyperbolicity implies Brody hyperbolicity. Brody [Br] discovered that for complex manifolds the converse holds under the additional condition of compactness. As is usual, we take the term hyperbolic to mean Kobayashi hyperbolic, and denote other hyperbolic properties by a prefix.

We shall sometimes need to consider hyperbolicity of subsets of almost complex manifolds; the definition is simply that the subset has a hyperbolic open neighborhood. In particular we will encounter hyperbolic manifolds with boundary, which for us will always be contained in a larger almost complex manifold. Thus the term compact manifold means possibly with boundary. Note that our terminology here is not consistent with standard terminology in the theory of complex manifolds. On the other hand, if a manifold is compact and has no boundary, we use the term closed manifold.

The following is a version of Brody's theorem [Br] for almost complex manifolds:

Theorem 2. **(i)** *Let (M^{2n}, J) be an almost complex manifold and $|\cdot|$ a continuous norm on TM. M is hyperbolic if and only if there exists a function $C : M \to \mathbb{R}$ (possibly a constant) bounded on compact sets in M and such that for all pseudoholomorphic mappings $f : \mathbb{D} \to M$ we have:*

$$|f_*(0)e| \leq C(f(0)).$$

(ii) *A compact almost complex manifold is hyperbolic if and only if it is Brody hyperbolic.*

Proof. Suppose that a function $C : M \to \mathbb{R}$ exists with $|f_*(0)e| \le C(f(0))$ for all pseudoholomorphic mappings $f : \mathbf{D} \to M$ and that C is bounded on compact sets. Then

$$|f_*(z)e| \le \frac{C(f(0))}{1 - |z|^2}$$

by precomposing with suitable disk automorphisms. Let ρ be the distance associated to $|\cdot|$. Then for every pseudoholomorphic mapping $f : \mathbf{D} \to M$ and for every pair $z_1, z_2 \in \mathbf{D}$ the following holds: $\rho(f(z_1), f(z_2)) \le C(f(0)) \cdot d(z_1, z_2)$. Now it is easy to see that for any two points $p \ne q$ with $R = \rho(p, q)$ sufficiently small, the last inequality implies $d_M(p, q) \ge \dfrac{R}{C_0} > 0$, $C_0 = \sup\limits_{p \in \bar{B}} C(p)$, where $\bar{B}$ is the ball $\bar{B}_\rho(p; R)$ with center p and radius R (one can assume that in the definition of a Kobayashi chain all the points $z_k = 0$). We note that it is sufficient to show that $d_M(p, q) > 0$ for p and q sufficiently close, by the triangle inequality. So the manifold M is hyperbolic.

Now assume that there exists no function $C(p)$ bounded on compact sets such that

$$|f_*(0)e| \le C(f(0))$$

for all pseudoholomorphic mappings $f : \mathbf{D} \to M$. Then there has to be some compact set K_0 on which such an inequality does not hold. For otherwise we can exhaust M by an ascending sequence K_j of compact subsets such that $|f_*(0)e|$ has a finite upper bound depending on j alone, valid for all pseudoholomorphic mappings $f : \mathbf{D} \to M$ with $f(0) \in K_j$. But clearly this yields a function $C : M \to \mathbb{R}$ as above.

So let $f_k : \mathbf{D} \to M$ be a sequence of pseudoholomorphic mappings with

$$|(f_k)_*(0)e| \to \infty,$$

and assume that all the points $f_k(0)$ lie in the compact set K_0. We will show that M is not hyperbolic. We can extract a subsequence to assure that $f_k(0) \to p \in M$. Let K be a compact set with p an interior point on which J is tamed by an exact symplectic form, and let r_k be the supremum of radii $r \le 1$ such that $f_k(r\mathbf{D}) \subseteq K$. Gromov's Schwarz lemma, see Corollary 4.1.4 of [Mu], implies that $r_k \to 0$, since $|(f_k)_*(0)e| \to \infty$. It is clear that there exists a sequence $z_k \in r_k\overline{\mathbf{D}}$ such that $q_k = f(z_k) \in \partial K$. Extract a subsequence to assure that $q_k \to q \in \partial K$. Then $d_M(p, q_k) \to d_M(p, q)$ by the continuity of d_M and

$$d_M(p, q_k) = d_M(f_k(0), f_k(z_k)) \leq d(0, z_k) \to 0$$

and so $d_M(p, q) = 0$, hence M is not hyperbolic.

It is clear that a hyperbolic almost complex manifold is Brody hyperbolic. Assume on the other hand that M is a compact, but not hyperbolic, almost complex manifold. We shall prove that there exists a nonconstant pseudoholomorphic mapping $g : \mathbb{C} \to M$.

We put a Riemannian metric $|\cdot|$ on M. Since M is not hyperbolic, there is a sequence of pseudoholomorphic mappings $f_k : \mathbb{D} \to M$ with $|(f_k)_*(0)e| \to \infty$. The proof of Brody's reparametrization lemma [Br] goes through unchanged for almost complex manifolds. We use it in the form that if (M^{2n}, J) is an almost complex manifold, $|\cdot|$ a continuous norm on TM and $f : r\mathbb{D} \to M$ a pseudoholomorphic mapping such that $|f_*(0)e| \geq c > 0$, then there exists a pseudoholomorphic mapping $h : r\mathbb{D} \to M$ such that $|h_*(0)e| = c/2$,

$$|h_*(z)e| \leq \frac{c}{2} \frac{r^2}{r^2 - |z|^2},$$

and $h(r\mathbb{D}) \subseteq f(r\mathbb{D})$. See p. 27 of [NO] for a proof that can be adapted virtually unchanged. We apply Brody's reparametrization lemma to the sequence of pseudoholomorphic mappings

$$f_k\left(\frac{2z}{|(f_k)_*(0)e|}\right)$$

from $r_k\mathbb{D}$ to M, where $r_k = |(f_k)_*(0)e|/2 \to \infty$. This gives a sequence of pseudoholomorphic mappings $g_k : r_k\mathbb{D} \to M$ with $|(g_k)_*(0)e| = 1$ and

$$|(g_k)_*(z)e| \leq \frac{r_k^2}{r_k^2 - |z|^2}.$$

Using the last inequality and the compactness theorem 4.1.3 of [MS], we conclude that g_k has a subsequence which converges uniformly with all derivatives on compact subsets of $\mathbb{C}$ to a pseudoholomorphic mapping $g : \mathbb{C} \to M$. It is nonconstant because $|(g)_*(0)e| = 1$. $\square$

We remark that the proof of theorem 2 is applicable to compact subsets of almost complex manifolds. Observe in particular that the proof of theorem 2 shows that a compact subset of an almost complex manifold is hyperbolic if and only if it is Brody hyperbolic.

More general versions of Brody's theorem are known for complex manifolds. See theorem 2.2 and 2.3 of Chap. III of [La]. They have analogs for almost complex manifolds, which are easily proved by modifying the proofs in [La] along the lines of the above proof. Then one can prove the following theorem in the same way as theorem 3.1 is proved in Chap. III of [La]:

Theorem 3. *Let E and B be subsets of almost complex manifolds and $\pi : E \to B$ a proper (restriction of a) pseudoholomorphic mapping. If B is hyperbolic and each fiber is a hyperbolic subset of E, then E is hyperbolic. If the fiber above a point of B is hyperbolic, this point has a neighborhood such that the fibers above all points of this neighborhood are hyperbolic.*

3. The automorphism group

In contrast to the complex case, generic almost complex structures admit no local automorphisms other than the identity and so the group $\mathrm{Aut}(M^{2n}, J)$ is usually trivial. The obstruction theory for local and formal automorphisms of almost complex manifolds was developed in [Kr1]. However in practice we usually need criteria for this generic property. Here we give one connected with the notion of hyperbolicity.

It is a result of Kobayashi [Ko1] that the automorphism group of a closed (usually called just compact in the literature but with the assumption of empty boundary) hyperbolic complex manifold is finite. This is also true in the almost complex case. For closed almost complex manifolds, the group of pseudoholomorphic diffeomorphisms of the manifold to itself is a Lie transformation group, when equipped with the topology of Σ-convergence, i.e. uniform convergence on compact sets of the mappings and their derivatives through the third order, as was proved by Boothby, Kobayashi and Wang [BKW].

Theorem 4. *The automorphism group $G = \mathrm{Aut}(M^{2n}, J)$ of a closed hyperbolic almost complex manifold (M^{2n}, J) is finite.*

Proof. Since G is a Lie group we can consider its Lie algebra $\mathcal{G}$ consisting of pseudoholomorphic vector fields; i.e. fields ξ on M^{2n} such that $L_\xi J = 0$ or

$[\xi, J\eta] = J[\xi, \eta]$ for every vector field η on M^{2n}. In particular, $[\xi, J\xi] = 0$. This means that if $0 \neq \xi \in \mathcal{G}$ then we have a nonconstant pseudoholomorphic curve $f : \mathbb{C} \to M^{2n}$ through a point $p \in M^{2n}$ of the form

$$z = x + iy \mapsto \exp(x\xi + y(J\xi))p.$$

The exponential mapping is globally defined since M^{2n} was assumed closed. Now (M^{2n}, J) was assumed hyperbolic and so such a nonconstant pseudoholomorphic curve cannot exist. Hence $\mathcal{G}$ is trivial and thus G is discrete.

We must now show that G is compact. Since M is compact and second countable, G is second countable. So it is enough to prove that any sequence $\varphi_k \in G$ has a Σ-convergent subsequence. In the following, each time we extract a subsequence, we give it the same notation as the original sequence. Note that G is a subgroup of the isometry group $\mathcal{I}(M, d_M)$ of M, where d_M is the Kobayashi distance on M. According to the theorem of van Dantzig and van der Waerden [DW], see also pp. 46-50 of [KN], $\mathcal{I}(M, d_M)$ is compact in the topology induced by d_M. Moreover this theorem easily generalizes to the following statement which we also use:

Lemma. *Let (A, d_A) and (B, d_B) be compact metric spaces. Let $\mathcal{I}(A, B)$ be the set of isometries $f : A \to B$, and topologize it by the compact-open topology. Then $\mathcal{I}(A, B)$ is compact.*

Let $\varphi_k \in G$ be a sequence, and extract a subsequence converging in $\mathcal{I}(M, d_M)$ to some $\varphi \in \mathcal{I}(M, d_M)$. Fix a Riemannian metric on M with corresponding norm $|\cdot|$. We use the boundness of derivatives from theorem 2 to deduce that the sequence $(\varphi_k)_*$ of first derivatives must be bounded. Otherwise we can extract a subsequence to assure that

$$|(\varphi_k)_*(p_k)v_k| \to \infty$$

where $|v_k| = 1$ and $v_k \in T_{p_k}M$. Now extract subsequences to assure that $p_k \to p \in M$ and $v_k \to v \in T_pM$, $|v| = 1$. Let ξ be a smooth vector field extending v and such that $|\xi| = 1$ on a neighborhood of p. By 5.4a of [NW] and arguments from §1, there exists a number $r > 0$ and a smooth family $f_q : r\mathbb{D} \to M$ for $q \in U$, where U is a neighborhood of p, such that $f_q(0) = q$ and $(f_q)_*(0)e = \xi_q$. Now we obtain a sequence of pseudoholomorphic mappings $\varphi_k \circ f_{p_k} : r\mathbb{D} \to M$ with the property that

$$|(\varphi_k \circ f_{p_k})_*(0)e| \to \infty,$$

which contradicts hyperbolicity.

Note also that we might obtain the boundness of the sequence of derivatives by the nonlinear Schwarz lemma of [Gr] using the compactness. Such a Schwarz lemma also holds for higher derivatives [Mu], which yields the desired convergence result.

Now consider the space $J^1_{PH}(M, M)$ of 1-jets of pseudoholomorphic mappings of M to itself, i.e. the set of points (p, q, Φ) with $p, q \in M$, $\Phi \in T^*_p M \otimes T_q M$, satisfying the equality $\Phi \circ J_p = J_q \circ \Phi$. This space carries a canonical almost complex structure [Gau] for which the standard projection $\pi : J^1_{PH}(M, M) \to M \times M$ is a pseudoholomorphic mapping. Let $\sigma : M \times M \to M$ be the projection on the second factor. Denote the composition by $\rho = \sigma \circ \pi$. Let $J^1_{PH}(M, M)_r$ denote the set of points $(p, q, \Phi) \in J^1_{PH}$ such that the element $\Phi \in T^*_p M \otimes T_q M$ which also could be considered as a tangent vector to J^1_{PH}, satisfies $|\Phi| \leq r$. Thus we have pseudoholomorphic mappings $\rho_r : J^1_{PH}(M, M)_r \to M$ and $\pi_r : J^1_{PH}(M, M)_r \to M \times M$. The compact ball $\pi_r^{-1}(p \times q)$ in the complex affine space $\pi^{-1}(p \times q)$ carries the standard complex structure and hence is hyperbolic for every p, $q \in M$ and M is also hyperbolic. The standard complex structure here means the restriction of the standard complex structure from an open neighborhood. We remind the reader that we call a subset of an almost complex manifold hyperbolic if it has a hyperbolic open neighborhood. Now by theorem 3 the compact set $J^1_{PH}(M, M)_r$ in $J^1_{PH}(M, M)$ is hyperbolic. For $\pi_r : J^1_{PH}(M, M)_r \to M \times M$ is clearly a proper mapping.

By the arguments above for some r we have the following commutative diagram of pseudoholomorphic mappings:

$$\begin{array}{ccc} & & J^1_{PH}(M, M)_r \\ & {}^{j^1\psi} \nearrow & \downarrow {}^{\rho_r} \\ M & \xrightarrow{\psi} & M. \end{array}$$

For $p, q \in M$ and an isometry $\psi \in G$ we have

$$d_M(p, q) \geq d_{(J^1_{PH})_r}(j^1\psi(p), j^1\psi(q)) \geq d_M(\psi(p), \psi(q)) = d_M(p, q),$$

and so for every $\psi \in G$, $j^1\psi$ is an isometry with respect to d_M and $d_{(J^1_{PH})_r}$. Using the generalization of the result of van Dantzig and van der Waerden quoted above, we conclude that we can extract a subsequence to assure that φ_k converges uniformly in J^1_{PH}, i.e. the sequence of derivatives converges uniformly. Exactly the same arguments above show that the derivatives of the mappings $j^1\varphi_k$ are bounded and we can consider the hyperbolic almost complex manifold $J^1_{PH}(M, J^1_{PH}(M,M)_r)_r$ to extract a subsequence for which we have uniform convergence in the C^2 sense. Applying the argument one more time, we obtain a Σ-convergent subsequence, thus $\varphi_k \to \varphi \in G$. $\square$

Theorem 5. *Let (M^{2n}, J) be a hyperbolic almost complex manifold.*
(i) $\dim \mathrm{Aut}(M^{2n}, J) \le 2n + n^2$ *and equality holds iff there is an isomorphism with the standard ball $(M^{2n}, J) \simeq (B^{2n}, J_0)$.*
(ii) *The isotropy subgroup* $\mathrm{Aut}(M^{2n}, J)_p$ *is compact for every $p \in M$.*
(iii) *No almost complex Lie group of positive dimension acts effectively as a pseudoholomorphic transformation group on (M^{2n}, J).*

Proof. This is an analog of theorem 9.1 of [Ko2], see also [Ko1]. The idea is the same with slight changes: we may modify the corresponding proof that G is a closed subgroup of $\mathcal{I}(M)$ using the ideas from the proof of theorem 4; we can avoid using the lemma and change $(J^1_{PH})_r$ to $(J^1_{PH})_C$, where $C : M \times M \to \mathbb{R}$ is some function bounded on compact sets and the PH-jets $\Phi \in T^*_pM \otimes T_qM$ satisfy $|\Phi| \le C(p,q)$. The third part is trivial: if vector fields $\xi, J\xi \in \mathcal{G}$, then they are complete and the exponential mapping is defined. $\square$

It turns out that the third statement of theorem 5 is still true for another wide class of almost complex manifolds. The following is a weak version of the general position property from [Kr1].

Definition. An almost complex manifold (M^{2n}, J) is of slightly general position if for a dense set of points $p \in M^{2n}$ the Nijenhuis tensor N_J at each of these points satisfies

$$\mathrm{Ker}[(N_J)_p(\xi, \cdot)] \ne T_pM^{2n}$$

for all vectors $\xi \in T_pM \backslash \{0\}$.

Proposition 2. *Let an almost complex manifold (M^{2n}, J) be of slightly general position. Then the Lie algebra $\mathcal{G}$ of the automorphism group $G = \mathrm{Aut}(M^{2n}, J)$ has the property $\mathcal{G} \cap (J\mathcal{G}) = 0$, i.e. the tangent space of G contains no complex lines: $\xi \in \mathcal{G} \Rightarrow J\xi \notin \mathcal{G}$.*

Proof. If $\xi \in \mathcal{G} = \mathrm{aut}(M^{2n}, J)$ then $[\xi, J\eta] = J[\xi, \eta]$ for any vector field η. If in addition $J\xi \in \mathcal{G}$ then $[J\xi, J\eta] = J[J\xi, \eta]$. These two equations give $N_J(\xi, \eta) = 0$ at p for every η and hence $\xi_p = 0$. Since the set of such points p is dense we have $\xi = 0$. $\square$

4. Examples

Numerous examples of hyperbolic and nonhyperbolic complex manifolds are considered in the literature. Here we consider examples with non-integrable almost complex structures.

Example 1. We start with an example of a nonhyperbolic almost complex manifold. The Kobayashi pseudodistance on a homogeneous almost complex manifold is invariant (for definitions and examples of homogeneous almost complex manifolds see [KN, Y]). Consider $S^6 = G_2/SU(3)$ with its well-known nonintegrable almost complex structure J that is defined by means of the octonions (or Cayley numbers). The definition (see [KN] for the full details) goes like this: let $\mathbb{R}^7 = \mathrm{Im}\,\boldsymbol{Ca}$ be the purely imaginary octonions and $S^6 \subset \mathbb{R}^7$ the unit sphere. On $\mathbb{R}^7$ there is defined a vector product $\times$ and we define $J : T_wS^6 \to T_wS^6$ by $\eta \mapsto \eta \times w$ where $\eta \in \mathbb{R}^7$ and $\eta \perp w$.

Now we show that $d_{S^6} \equiv 0$. Let us take arbitrary $\eta \in T_wS^6$ such that $|w| = |\eta| = 1$, $\eta \perp w$ and set $\nu = \eta \times w$. The alternativity law implies that $\mathbb{R}^3 = \langle \eta, w, \nu \rangle \subset \boldsymbol{Ca}$ is an invariant under the vector product $\times$ subspace which is isomorphic to the standard $(\mathbb{R}^3 = \langle i, j, k \rangle, \times)$. Thus $S^2 = \mathbb{R}^3 \cap S^6$ is J-invariant, i.e. pseudoholomorphic. Now the claim follows since for every two points $w_1, w_2 \in S^6$ we can find $\eta \in T_{w_1}S^6$ such that the corresponding sphere S^2 goes through w_2.

Example 2. An almost complex structure J is called tame if there exists a symplectic form ω such that $\omega(X, JX) > 0$ for any nonzero vector X, see [Gr]. For tame almost complex structures on bounded domains in $\mathbb{R}^{2n}$ we have a sufficient condition for hyperbolicity.

Theorem 6. *Let (D^{2n}, J) be an almost complex domain with J tame. If there exists a ball $D' \supset\supset D$ such that J is the restriction of some tame almost complex structure on D', then (D^{2n}, J) is hyperbolic.*

Proof. It was shown by Lafontaine [Laf] that if D is a bounded domain in $(\mathbb{R}^{2n}, J)$, where the almost complex structure J is tamed by a symplectic form with bounded coefficients, then (D^{2n}, J) is Brody hyperbolic. We can suppose the coefficients of the taming symplectic form on D' are bounded, shrinking this ball a little otherwise. Now consider a diffeomorphism of D' onto $\mathbb{R}^{2n}$ which maps D onto a bounded domain. The coefficients of the transported symplectic form remain bounded. Thus we have exhibited D as a bounded domain in a tame $(\mathbb{R}^{2n}, J')$, since the diffeomorphism does not disturb the taming condition. Then the closure of D is Brody hyperbolic since it is contained in a slightly bigger open ball. Now apply the remark following the proof of theorem 2 to the closure of D. $\square$

The above result is the analog of the known sufficient condition for a domain in $\mathbb{C}^n$ to be hyperbolic; that it be bounded. For this can be formulated as the statement that a bounded domain in $\mathbb{R}^{2n}$ with complex structure is hyperbolic if the complex structure can be extended to the standard complex structure on an ambient ball. A statement of this kind is however false for a general almost complex domain where some extra condition such as tameness or standard integrability is not imposed. For $S^6\backslash\{p_0\}$ is diffeomorphic to $\mathbb{R}^6$ so we can essentially impose on $\mathbb{R}^6$ the restriction of the almost complex structure on S^6 discussed above. Choosing p_0 not to lie on the pseudoholomorphic 2-sphere that we found, and removing a closed ball around p_0, we obtain a bounded domain in $\mathbb{R}^6$ (actually an open ball), such that the almost complex structure extends to the whole of $\mathbb{R}^6$. Yet this domain is not hyperbolic.

Note that the reasoning above shows that the almost complex structure on S^6 cannot be tamed even on the complement of a small closed ball (globally it is evident since S^6 is not symplectic: $H^2(S^6) = 0$).

Example 3. In Example 2 we saw that not every bounded domain in an almost complex $\mathbb{R}^{2n}$ with $n > 1$ is hyperbolic (except maybe for $\mathbb{R}^4$). However theorem 6 implies that every point possesses a neighborhood which is hyperbolic. Thus it follows from theorem 3 that for every almost complex manifold M with finite dimensional (C^∞-)parametrized almost complex structure $J^{(\tau)}$, $\tau \in \mathbb{R}^p$, the property of $(M, J^{(\tau)})$ being hyperbolic is open with respect to (the usual) topology on $\mathbb{R}^p$. This can be generalized.

The following statement gives examples of non-integrable hyperbolic almost complex structures on compact manifolds, for example on the product of closed Riemann surfaces each of genus greater than one or on a closed Hermitian manifold with negative holomorphic sectional curvature.

Theorem 7. *Let (M, J) be a compact hyperbolic almost complex manifold. Then for a small neighborhood $\mathcal{O}$ of J in the C^∞ topology every almost complex manifold (M, J') with $J' \in \mathcal{O}$ is hyperbolic.*

Proof. Assume that the statement is false. Then for every sequence of neighborhoods $\mathcal{O}_k$ of J shrinking to J there exists a sequence $J_k \in \mathcal{O}_k$ of almost complex structures such that the manifold (M, J_k) is not hyperbolic. For example one can take a sequence of neighborhoods $\mathcal{O}_k^{(m)}$ to be the ball of radius $\frac{1}{k}$ around J in C^m-norm (or $W^{m,p}$-norm), intersect it with $\mathcal{O}$ and then take the diagonal subsequence. According to theorem 2 for every J_k there exists a nonconstant pseudoholomorphic curve $f_k : \mathbb{C} \to (M, J_k)$. The sequence of almost complex structures J_k tends to J in the C^∞ topology. As in the proof of theorem 2 we can apply Brody's reparametrization lemma to obtain uniform boundness of derivatives on compact sets and the condition $|(f_k)_*(0)e| = 1$, and thus due to the compactness theorem B.4.2. of [MS] there exists a subsequence converging uniformly with all derivatives on compact sets to some pseudoholomorphic curve $f : \mathbb{C} \to (M, J)$. Since $|f_*(0)e| = 1$ this curve is nonconstant which contradicts the hyperbolicity of (M, J). $\square$

Example 4. In the paper [Ban] Bangert proved that almost complex tori (T^{2n}, J) with J taming the standard symplectic structure ω_0 admit nonconstant pseudoholomorphic curves $f : \mathbb{C} \to T^{2n}$. Here we sketch another proof of this fact.

Consider the universal covering $\mathbb{R}^{2n}$ of T^{2n} and denote the almost complex structure induced on $\mathbb{R}^{2n}$ via the covering projection by J. Compactify $\mathbb{R}^{2n}$ in a different way from Bangert's: change the almost complex structure outside a big polydisk centered at the origin to become integrable and glue the neighborhood to the product $M^{2n} = S^2 \times S^2 \times \ldots \times S^2$ (cf. the construction from proposition 2 of [Kr2]). Denote the constructed almost complex structure on M^{2n} by $\tilde{J}$.

The structure $\tilde{J}$ can be perturbed to become a structure of general position J_ν. Consider the homology class A_k of the k-th sphere-factor of $M^{2n} = S^2 \times S^2 \times \ldots \times S^2$. By definition it is ω_M-simple for the standard symplectic structure ω_M on M^{2n}. Thus calculating the evaluation map (see

[MS]) we conclude that there exists a J_ν-holomorphic A_k-sphere through every point of M^{2n}. Choose such a sphere through the origin, and precompose with a complex-affine mapping of S^2 onto itself, so that the J_ν-holomorphic map of the sphere into (M^{2n}, J_ν) is normalized at the origin. So we have arranged that the origin goes to the origin and the derivative at the origin has norm 1. Now let a sequence J_ν of almost complex structures of general position tend to $\tilde{J}$. By Gromov's compactness theorem [Gr] the corresponding sequence of J_ν-holomorphic spheres $u_\nu : S^2 \to M^{2n}$ in a fixed homology class A_k has a subsequence converging to some $\tilde{J}$-holomorphic sphere $u : S^2 \to M^{2n}$ in the same homology class. Since the polydisk from which we glue M^{2n} can be chosen as large as one desires the construction yields normalized J-holomorphic disks $f_r : r\mathbb{D} \to (\mathbb{R}^{2n}, J)$ with r as large as we please. Postcomposing with the covering projection yields normalized J-holomorphic disks $g_r : r\mathbb{D} \to (T^{2n}, J)$, so (T^{2n}, J) is not hyperbolic. Since it is compact, it cannot be Brody-hyperbolic, so there exists a nonconstant pseudoholomorphic mapping $g : \mathbb{C} \to T^{2n}$.

Note that actually this method leads to a generalization of Bangert's result giving n distinct entire lines $g_k : \mathbb{C} \to T^{2n}$ through a given point. Moreover a further generalization is connected with the fact that we can take the C^∞-closure of the set of all tame almost complex structures. This follows from theorem 7.

Another approach to get an almost complex structure of general position on M^{2n} is based on theorem 7: if we assume hyperbolicity of the almost complex structure on the torus T^{2n} it can be perturbed into general position with preservation of the hyperbolicity. But then we would need to control our construction in order that this general position property not be destroyed under the gluing.

Note also that in dimension $2n = 4$ Bangert's result follows for a subset of all almost complex structures from the paper by Kuksin [Ku]. Actually he proved that generic almost complex structures are divided into two classes: one has degree equal to an even nonzero number and the other equal to zero. Here degree means the index of the corresponding elliptic operator as in the Gromov theory. The nonvanishing of index is equivalent to the existence of at least deg = ind tori T^2 in a given homology class. Thus for the first class the existence of entire lines follows. The second case remains unclear with this approach.

5. Concluding remarks

Note that the Kobayashi-Royden infinitesimal pseudometric F_M may also be defined on almost complex manifolds using the results of Nijenhuis and Woolf. The definition is

$$F_M(p;v) = \inf \frac{1}{r}$$

where $p \in M$, $v \in T_pM$ and the infimum is taken with respect to all pseudoholomorphic mappings $f : \mathbb{D} \to M$ with $f(0) = p$ and $f_*(0)e = rv$. This may be reformulated as another invariant $\pi F_M(p;v)^{-2} = \sup(\pi r^2)$ which looks similar to the so-called Gromov width from symplectic geometry, see [Gr, MS]. The pseudometric F_M and the corresponding geometry of pseudoholomorphic disks are studied in [Kr2].

On a Kähler manifold one can consider further invariants coming from the Kobayashi-Royden pseudometric and the Riemannian metric. As follows from our theorem 1 this is also the case for almost Kähler manifolds. For example we consider the following functions on almost Kähler manifold (M, g, ω, j) with values from $[0, +\infty]$:

$$\rho^+_{M,j}(x) = \inf_{\xi \in T_xM \setminus \{0\}} \frac{|\xi|_g}{F_M(\xi)}, \quad \rho^-_{M,j}(x) = \sup_{\xi \in T_xM \setminus \{0\}} \frac{|\xi|_g}{F_M(\xi)}.$$

Note that the standard methods of generating symplectic invariants via almost complex ones are not working here. Actually: $\sup_{j \in \mathcal{T}(\omega)} \rho^+_{M,j}(x) = +\infty$, $\inf_{j \in \mathcal{T}(\omega)} \rho^-_{M,j}(x) = 0$. Here $\mathcal{T}(\omega)$ is the set of all almost complex structures compatible with the symplectic form ω on M, i.e. such structures j that the manifold M become almost Kähler with the metric $g(\xi, \eta) = \omega(\xi, j\eta)$.

References

[Ban] V. Bangert, *"Existence of a complex line in tame almost complex tori"*, *Duke Math. J.*, **94**, no. 1 (1998), 29–40

[Ba] T. J. Barth, *"The Kobayashi distance induces the standard topology"*, *Proc. Amer. Math. Soc.*, **35**, no. 2 (1972), 439–440

[BKW] W. M. Boothby, S. Kobayashi, H. C. Wang, *"A note on mappings and automorphisms of almost complex manifolds"*, *Ann. Math.*, **77** (1963), 329–334

[Br] R. Brody, *"Compact manifolds and hyperbolicity"*, *Trans. Amer. Math. Soc.*, **235** (1978), 213–219

[DW] D. van Dantzig, B.L. van der Waerden *"Über metrisch homogenen Räume"*, *Abh. Math. Sem. Hamburg*, **6** (1928), 374–376

[Gau] P. Gauduchon, *"The canonical almost complex structure on the manifold of 1-jets of pseudo-holomorphic mappings between two almost complex manifolds"*, appendix to chap. III, pp. 69–73, of M. Audin, J. Lafontaine Eds. *"Holomorphic curves in symplectic geometry"*, Birkhäuser Verlag, Progress in Mathematics, **117** (1994)

[Gr] M. Gromov, *"Pseudo-holomorphic curves in symplectic manifolds"*, *Invent. Math.*, **82** (1985), 307–347

[JP] M. Jarnicki, P. Pflug, *"Invariant Distances and Metrics in Complex Analysis"*, de Gruyter, (1993)

[KN] S. Kobayashi, K. Nomizu, *"Foundations of Differential Geometry"*, Wiley-Interscience, (1963)

[Ko1] S. Kobayashi, *"Hyperbolic manifolds and holomorphic mappings"*, Marcel Dekker, (1970)

[Ko2] S. Kobayashi, *"Intrinsic distances, measures, and geometric function theory"*, *Bull. Amer. Math. Soc.*, **82** (1976), 357–416

[KO] B. Kruglikov, M. Overholt, *"The Kobayashi pseudodistance on almost complex manifolds"*, prepr. Univ. Tromsoe, 97-19 (1997); e-print: xxx.lanl.gov/abs/dg-ga/9703005

[Kr1] B. Kruglikov, *"Nijenhuis tensors and obstructions for pseudo-holomorphic mapping constructions"*, prepr. Univ. Tromsoe, 96-16 (1996); e-print: xxx.lanl.gov/abs/dg-ga/9611004

[Kr2] B. Kruglikov, *"Existence of close pseudoholomorphic disks for almost complex manifolds and application to the Kobayashi-Royden pseudonorm"* , Functional Analysis and Applications, to appear

[Ku] S. B. Kuksin, *"Pseudoholomorphic 2-tori in T^4"* , *Differential Geom. Appl.*, **6** (1996), no. 2, 109-119

[Laf] J. Lafontaine, *"A non-linear Hadamard theorem"*, *Symplectic geometry, groupoids, and integrable systems (Berkeley 1989), 189–192*, Math. Sci. Res. Inst. Publ., **20**, Springer (1991)

[La] S. Lang, *"Introduction to Complex Hyperbolic Spaces"*, Springer, (1987)

[Mu] M. P. Muller, *"Gromov's Schwarz lemma"*, M. Audin, J. Lafontaine Eds. *"Holomorphic curves in symplectic geometry"*, Birkhäuser Verlag, Progress in Mathematics, **117** (1994)

[MS] D. McDuff, D. Salomon, *"J-holomorphic curves and Quantum cohomology"*, AMS, *University Lecture Series*, **6** (1994)

[NW] A. Nijenhuis, W. Woolf, *"Some integration problems in almost-complex and complex manifolds"*, *Ann. Math.*, **77** (1963), 424–489

[NO] J. Noguchi, T. Ochiai, *"Geometric Function Theory in Several Complex Variables"*, AMS, *Translations of Mathematical Monographs*, **19** (1990)

[Pi] G. Pick, *"Über eine Eigenschaft der konformen Abbildung kreisförmiger Bereiche"*, *Math. Ann.*, **77** (1915), 1–6

[Y] K. Yang, *"Almost Complex Homogeneous Spaces and their Submanifolds"*, World Scientific, (1987)

- Math. Modelling Chair, Moscow State Techn. Univ. n. a. Baumann, Moscow, Russia
  kruglikov@glasnet.ru
- Dept. of Mathematics and Statistics, Univ. of Tromso, N-9037, Tromso, Norway
  marius@math.uit.no