PRODUCTS AND QUANTIZATIONS IN K-THEORY OF BRAIDING-COMMUTATIVE ALGEBRAS
(Dedicated to the memory of Robert Wayne Thomason)

VALENTIN LYCHAGIN, ANDREI V. PRASOLOV
Institute of Mathematics and Statistics, University of Tromsø,
N-9037, Tromsø, Norway
E-mail: lychagin@math.uit.no, prasolov@math.uit.no

In the paper we outline an approach to investigate multiplicative structures in K-theory of braiding-commutative algebras. We introduce quantizations of underlying monoidal categories which induce quantizations of the multiplicative structure on the corresponding K-theory.

1 Introduction

[abc]In this paper we investigate multiplications in K-theory of algebras which are commutative with respect to some braiding (or a solution of the Yang - Baxter equation). As far as the authors are aware of, multiplicative structures on K-theory of a ring are known only for the case of commutative ones.

We show that under natural conditions such multiplicative structures exist for extremely wide class of algebras.

The paper is organized as follows. At the beginning of the first section we recall main facts and constructions for monoidal categories. We pay a special attention to the natural notion of quantization in monoidal categories [7] which gives us possibility to easily change algebras and modules under consideration. We introduce also the notions of σ-symmetric modules and σ-commutative algebras. These notions are basic in the construction of calculus and differential equations in monoidal categories (so called quantum differential equations) [7]. We show also that symmetric modules produce a new monoidal category which admits a braiding induced by the "old" one (Theorem 2.30).

In the K-theory of a σ-commutative algebra A, the "new" tensor product $\otimes_A$ induces a ring structure on $K_*(A)$, while symmetric modules generate a subring of the graded center $K_*(A)$ (Theorem 3.8). We investigate this ring structure in the following cases: algebras over a quasi-triangular Hopf algebra (Theorem 4.3), G-graded algebras (Theorem 4.6), superalgebras (Theorem 4.9) and twisted G-algebras (Theorem 4.11).

[a] *1991 Mathematics Subject Classification:* Primary 19D50, 19D23; Secondary 18D10.
[b] *Key words and phrases:* Monoidal category, braiding, symmetric module, braiding-commutative algebra, quantization, K-theory, G-theory, K-functors, G-functors, products.
[c] The authors would express their gratitude to the creator of "diagrams.tex" Paul Taylor.

2 Modules and quantizations in monoidal categories

We begin with main properties of categories under consideration. We shall denote by $\mathcal{O}b\,(\mathfrak{C})$ class of objects in category $\mathfrak{C}$ and by $\mathcal{M}or\,(X,Y)$ the set of morphisms from X to Y.

2.1 Monoidal categories

Let $\mathbf{k}$ be a commutative ring with unit and let $\mathfrak{C}$ be a $\mathbf{k}$-linear category. The category is said to be *monoidal* if $\mathfrak{C}$ is equipped with

1. Tensor product $\mathbf{k}$-bilinear covariant functor $\otimes : \mathfrak{C}\times\mathfrak{C}\longrightarrow\mathfrak{C}$

   where
   $$\otimes : X\times Y \longmapsto X\otimes Y, X, Y\in Ob\,(\mathfrak{C})\,,$$
   and
   $$\otimes : \phi\times\psi\longmapsto\phi\otimes\psi$$
   if $\phi,\psi\in\mathcal{M}or\,(\mathfrak{C})\,,$

2. A natural $\mathbf{k}$-linear isomorphism
   $$\alpha = \alpha_{X,Y,Z} : X\otimes(Y\otimes Z)\longrightarrow(X\otimes Y)\otimes Z$$
   which is called the *associativity constraint,*

A *unit object* $\mathbf{1}$ with natural isomorphisms
$$\eta^r : X\otimes\mathbf{1}\longrightarrow X \quad\text{and}\quad \eta^l : \mathbf{1}\otimes X\longrightarrow X$$

such that the MacLane coherence conditions hold [11]:

✠ (the *pentagon axiom*)

$$\begin{tikzcd}
X\otimes(Y\otimes(Z\otimes T)) \arrow[r, "\alpha_{X,Y,Z\otimes T}"] \arrow[d, "id_X\otimes\alpha_{Y,Z,T}"] & (X\otimes Y)\otimes(Z\otimes T) \arrow[r, "\alpha_{X\otimes Y,Z,T}"] & ((X\otimes Y)\otimes Z)\otimes T \\
X\otimes((Y\otimes Z)\otimes T) \arrow[rr, "\alpha_{X,Y\otimes Z,T}"] & & (X\otimes(Y\otimes Z))\otimes T \arrow[u, "\alpha_{X,Y,Z}\otimes id_T"']
\end{tikzcd}$$

✠✠ (the *unity axiom*)

$$
\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "\alpha_{X,1,Y}"] \arrow[dr, "\eta^r \otimes id_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "id_X \otimes \eta^l"] \\
& X \otimes Y &
\end{tikzcd}
$$

Moreover, due to the MacLane theorem, we shall assume that $\mathfrak{C}$ is a *strict category,* that is the associativity and unity constraints are identity morphisms.

Example 2.1 (Main) *Let H be a Hopf $\mathbf{k}$-algebra with a diagonal $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to \mathbf{k}$ and antipode $S : H \to H$. Let $\mathfrak{C} = H$-$\mathcal{M}$od be the category of left H-modules. Morphisms in the category are H-module homomorphisms. To convert $\mathfrak{C}$ into a monoidal category we define a bifunctor of tensor product to be the usual tensor product of modules over $\mathbf{k}$:*

$$X \otimes Y \overset{def}{=} X \underset{\mathbf{k}}{\otimes} Y.$$

Define an H-module structure on the tensor product as follows

$$h\left(x \otimes y\right) = \sum_i h_i^{(1)}\left(x\right) \otimes h_i^{(2)}\left(y\right)$$

where $x \in X$, $y \in Y$, and

$$\Delta\left(h\right) = \sum_i h_i^{(1)} \otimes h_i^{(2)}$$

in a slight modification of the Sweedler notation.

The next examples are specializations of the main one.

Example 2.2 (G-graded modules) *Let G be a finite multiplicative group and $H = \mathbf{k}\left(G\right)$ be the $\mathbf{k}$-algebra of functions on G with values in $\mathbf{k}$. The Hopf structure on $\mathbf{k}\left(G\right)$ is given by the usual coproduct, counit and antipode:*

$$\Delta\left(f\right)\left(g_1, g_2\right) = f\left(g_1 g_2\right), \ \varepsilon\left(f\right) = f\left(e\right), \ S\left(f\right)\left(g\right) = f\left(g^{-1}\right)$$

where $f \in \mathbf{k}\left(G\right)$, $g_1, g_2 \in G$, and we identify $\mathbf{k}\left(G\right) \otimes \mathbf{k}\left(G\right)$ with $\mathbf{k}\left(G \times G\right)$. Denote by θ_g the basis in $\mathbf{k}\left(G\right)$; $\theta_g\left(g'\right) = 0$ if $g' \neq g$ and $\theta_g\left(g\right) = 1$. In terms *of this basis the Hopf structure in* $\mathbf{k}(G)$ *has the form:*

$$\begin{aligned}
&\theta_g \cdot \theta_g = \theta_g, \quad \text{and} \quad \theta_g \cdot \theta_{g'} = 0 \quad \text{if } g' \neq g, \\
&\Delta(\theta_g) = \sum_{h \in G} \theta_h \otimes \theta_{h^{-1}g}, \\
&\varepsilon(\theta_g) = 0 \quad \text{if } g \neq e, \text{ and } \varepsilon(\theta_e) = 1, \\
&S(\theta_g) = \theta_{g^{-1}}.
\end{aligned}$$

The first property shows that θ_g *are projectors and therefore the category* $\mathbf{k}(G)$-$\mathcal{M}$od *coincides with category* G-$\mathcal{G}$rad *of* G*-graded modules,* $X = \coprod_{g \in G} X_g$*, equipped with the following* $\mathbf{k}(G)$*-action:*

$$\theta_g \left(\sum_{h \in G} x_h \right) = x_g.$$

Morphisms $f : X \to Y$ *in the category are* G*-graded* $\mathbf{k}$*-morphisms:* $f(X_g) \subset Y_g$ *for all* $g \in G$*.*

Tensor product in the category is the usual G*-graded tensor product:*

$$(X \otimes Y)_g = \sum_{h \in G} X_h \otimes Y_{h^{-1}g}.$$

Example 2.3 (G-modules) *As above let* G *be a finite group and let* $H = \mathbf{k}[G]$ *be the group algebra,* $\mathbf{k}[G] = (\mathbf{k}(G))^*$. *The Hopf structure in terms of the dual basis of* δ*-functions* δ_g, $g \in G$*, has the form:*

$$\delta_g \cdot \delta_h = \delta_{gh}, \ \Delta(\delta_g) = \delta_g \otimes \delta_g, \ S(\delta_g) = \delta_{g^{-1}}, \ \varepsilon(\delta_g) = 1.$$

The category of H*-modules is the category* G-$\mathcal{M}$od *of left* G*-modules over* $\mathbf{k}$.

Example 2.4 (Color modules) *[9] Let* $H = \mathbf{k}(G) \# \mathbf{k}[G]$ *be the smash product or double of* $\mathbf{k}(G)$*, i.e. the Hopf algebra generated by* θ_g *and* δ_g, $g \in G$*, and relations* $\delta_h \bullet \theta_g = \theta_{hgh^{-1}} \delta_h$*. The category of* H*-modules is a category* $\mathcal{C}_G$ *of* G*-graded* $\mathbf{k}$*-modules* $X = \sum_{g \in G} X_g$ *equipped with* G*-action, such that*

$$h(X_g) \subset X_{hgh^{-1}}.$$

2.2 Braidings

A *braiding* [4] in a monoidal $\mathbf{k}$-linear category is a natural $\mathbf{k}$-bilinear isomorphism $\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$, called also commutativity constraint, such that the hexagon diagrams

$$
\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[dl, "id_X \otimes \sigma_{Y,Z}"'] \arrow[dr, "\alpha_{X,Y,Z}"] & \\
X \otimes (Z \otimes Y) \arrow[d, "\alpha_{X,Z,Y}"'] & & (X \otimes Y) \otimes Z \arrow[d, "\sigma_{X \otimes Y, Z}"] \\
X \otimes (Z \otimes Y) \arrow[dr, "\sigma_{X,Z} \otimes id_Y"'] & & Z \otimes (X \otimes Y) \arrow[dl, "\alpha_{Z,X,Y}"] \\
& (Z \otimes X) \otimes Y &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[dl, "\sigma_{X,Y} \otimes id_Z"'] \arrow[dr, "\alpha^{-1}_{X,Y,Z}"] & \\
(Y \otimes X) \otimes Z \arrow[d, "\alpha^{-1}_{Y,X,Z}"'] & & X \otimes (Y \otimes Z) \arrow[d, "\sigma_{X, Y \otimes Z}"] \\
Y \otimes (X \otimes Z) \arrow[dr, "id_Y \otimes \sigma_{X,Z}"'] & & (Y \otimes Z) \otimes X \arrow[dl, "\alpha^{-1}_{Y,Z,X}"] \\
& Y \otimes (Z \otimes X) &
\end{tikzcd}
$$

and

$$\begin{tikzcd}
1 \otimes X \arrow[rr, "\sigma"] \arrow[dr, "\eta^l"'] & & X \otimes 1 \arrow[dl, "\eta^r"] \\
& X &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes 1 \arrow[rr, "\sigma"] \arrow[dr, "\eta^r"'] & & 1 \otimes X \arrow[dl, "\eta^l"] \\
& X &
\end{tikzcd}$$

commute for all $X, Y, Z \in \mathcal{O}b(\mathfrak{C})$.

In the case of strict monoidal categories the hexagon diagrams take the following form

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\sigma_{X \otimes Y, Z}"] \arrow[dr, "id_X \otimes \sigma_{Y,Z}"'] & & Z \otimes X \otimes Y \arrow[dl, "\sigma_{X,Y} \otimes id_Z"] \\
& X \otimes Z \otimes Y &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\sigma_{X \otimes Y, Z}"] \arrow[dr, "\sigma_{X,Y} \otimes id_Z"'] & & Z \otimes X \otimes Y \arrow[dl, "id_Y \otimes \sigma_{X,Z}"] \\
& Y \otimes X \otimes Z &
\end{tikzcd}$$

(see, for example, [11], [4]).

In the case when $\sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in Ob\,(\mathfrak{C})$ one needs only one, say, the first hexagon diagram. Such braidings are called *symmetries.*

In what follows a *braided category* means a strict monoidal $\mathbf{k}$-linear category $\mathfrak{C}$ equipped with a braiding.

The hexagon conditions yield the commutativity of the following *Yang - Baxter hexagon:*

$$\begin{tikzcd}
Y \otimes X \otimes Z \arrow[d, "id_Y \otimes \sigma_{X,Z}"'] & X \otimes Y \otimes Z \arrow[l, "\sigma_{X,Y} \otimes id_Z"'] \arrow[r, "id_X \otimes \sigma_{Y,Z}"] & X \otimes Z \otimes Y \arrow[d, "\sigma_{X,Z} \otimes id_Y"] \\
Y \otimes Z \otimes X \arrow[r, "\sigma_{Y,Z} \otimes id_X"] & Z \otimes Y \otimes X & Z \otimes X \otimes Y \arrow[l, "id_Z \otimes \sigma_{X,Y}"']
\end{tikzcd}$$

Denote by $\mathfrak{Y}_{X,Y,Z} : X \otimes Y \otimes Z \to Z \otimes Y \otimes X$ the composition of morphisms from the left half of the hexagon.

Example 2.5 (main) *Any braiding in the category is given by a* braiding element $\sigma \in H \otimes H$ *such that*

$$\begin{aligned}
(\mathrm{id}_H \otimes \Delta)(\sigma) &= (\sigma \otimes 1) \cdot \sigma_{13}, \\
(\Delta \otimes \mathrm{id}_H)(\sigma) &= (1 \otimes \sigma) \cdot \sigma_{13}, \\
\sigma \cdot \tau(\Delta(h)) &= \Delta(h) \cdot \sigma
\end{aligned}$$

for all $h \in H$.

Here $\tau : H \otimes H \to H \otimes H$ *is the natural twist* $\tau(a \otimes b) = b \otimes a$, *and* $\sigma_{13} = (\mathrm{id}_H \otimes \tau)(\sigma \otimes 1)$.

The corresponding braiding $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ *acts as following*

$$\sigma_{X,Y}(x \otimes y) = \sum \sigma'(y) \otimes \sigma''(x)$$

if $\sigma = \sum \sigma' \otimes \sigma''$.

Hopf algebras H *equipped with a braiding element* $\sigma \in H \otimes H$ *are called quasi-triangular.*

Example 2.6 (G-graded modules) *A braiding element* $\sigma \in \mathbf{k}(G) \otimes \mathbf{k}(G)$ *can be written down as*

$$\sigma = \sum_{a,b \in G} s(a,b)\, \theta_a \otimes \theta_b.$$

From the properties of the braiding elements it follows that

1. *G is an Abelian group, and*

2. *$s : G \times G \to U(\mathbf{k})$ is a group bihomomorphism to the group of units.*

3. *The braiding is a symmetry iff the following* **multiplicative skew symmetry** *property holds:*

$$s(a,b) \cdot s(b,a) = 1.$$

Example 2.7 (G-modules) $\mathbf{k} = \mathbb{C}$. *A braiding element* $\sigma = \sum_{a,b\in G} s(a,b)\,\delta_a \otimes \delta_b$ *can be considered as a* $\mathbb{C}$ *- linear map :*

$$\widehat{\sigma} : \mathbb{C}(G) \longrightarrow \mathbb{C}[G]$$

where $\widehat{\sigma}(\theta_a) = \sum_b s(a,b)\,\delta_b$.

In terms of the maps braiding elements admit the following description [5]. Let us fix two central subgroups $H_1, H_2 \subset G$ *and group homomorphism* $\phi : \widehat{H_2} \to H_1$, *where* $\widehat{H_2}$ *is the dual group for* H_2. *Then any braiding element can be obtained from the following commutative diagram*

$$\begin{tikzcd} \mathbb{C}(G) \arrow[r, "\widehat{\sigma}"] \arrow[d, "r_1"] & \mathbb{C}[G] \\ \mathbb{C}(H_1) \arrow[r, "\widetilde{\phi}"] & \mathbb{C}[H_2] \arrow[u, "{r_2}^*"'] \end{tikzcd}$$

where $r_i : \mathbb{C}(G) \to \mathbb{C}(H_i)$ *are the restriction maps, and* $\widetilde{\phi}$ *is the composition* $\widetilde{\phi} = \mathcal{F} \circ \phi^*$ *of* $\phi^* : \mathbb{C}(H_1) \to \mathbb{C}\left(\widehat{H_2}\right)$ *and the Fourier transform* $\mathcal{F} : \mathbb{C}\left(\widehat{H_2}\right) \to \mathbb{C}[H_2]$.

Example 2.8 (Color modules) *Braiding elements have the form [9]*

$$\sigma = \sum_{a,b\in G} s(a,b)\,\delta_a \theta_b \otimes \theta_b$$

and the corresponding braiding acts as follows

$$\sigma_{X,Y}(x_a \otimes y_b) = s(a,b)\,a(y_b) \otimes x_a$$

where function $s : G \times G \to U(\mathbf{k})$, *called* **color on** G, *satisfies the following conditions*

$$\begin{aligned} s\left(gag^{-1}, gbg^{-1}\right) &= s(a,b), \\ s(a,bc) &= s(a,b)\,s(a,c), \\ s(ab,c) &= s\left(a, bcb^{-1}\right) s(b,c) \end{aligned}$$

for all $a,b,c,g \in G$.

2.3 Quantizations

Let $\mathfrak{C}$ and $\mathfrak{C}'$ be $\mathbf{k}$-linear monoidal categories and let $\Phi : \mathfrak{C} \longrightarrow \mathfrak{C}'$ be a $\mathbf{k}$-linear functor which preserves units, $\Phi\left(1_{\mathfrak{C}}\right) = 1_{\mathfrak{C}'}$. We say (see [8]) that a natural $\mathbf{k}$-bilinear transformation $\mathcal{Q}_{X,Y} : \Phi(X) \otimes \Phi(Y) \longrightarrow \Phi(X \otimes Y)$, $X, Y \in Ob\left(\mathfrak{C}\right)$ is a *quantization of the functor* Φ if $\Phi\left(\eta^r_X\right) \circ \mathcal{Q}_{\mathbf{1},X} = \eta^r_{\Phi(X)}$, $\Phi\left(\eta^l_X\right) \circ \mathcal{Q}_{X,\mathbf{1}} = \eta^l_{\Phi(X)}$ and the following diagram

$$\begin{tikzcd}
\Phi(X) \otimes \Phi(Y) \otimes \Phi(Z) \arrow[r, "id_{\Phi(X)} \otimes \mathcal{Q}_{Y,Z}"] \arrow[d, "\mathcal{Q}_{X,Y} \otimes id_{\Phi(Z)}"] & \Phi(X) \otimes \Phi(X \otimes Z) \arrow[d, "\mathcal{Q}_{X,Y \otimes Z}"] \\
\Phi(X \otimes Y) \otimes \Phi(Z) \arrow[r, "\mathcal{Q}_{X \otimes Y,Z}"] & \Phi(X \otimes Y \otimes Z)
\end{tikzcd}$$

commutes.

We say that quantization $\mathcal{Q}$ is *regular* if $\mathcal{Q}$ is a natural isomorphism.

In general categories the last diagram has the form

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "id_X \otimes \mathcal{Q}_{Y,Z}"] \arrow[d, "\alpha_{X,Y,Z}"] & X \otimes (Y \otimes Z) \arrow[r, "\mathcal{Q}_{X,Y \otimes Z}"] & X \otimes (Y \otimes Z) \arrow[d, "\alpha_{X,Y,Z}"] \\
(X \otimes Y) \otimes Z \arrow[r, "\mathcal{Q}_{X,Y} \otimes id_Z"] & (X \otimes Y) \otimes Z \arrow[r, "\mathcal{Q}_{X \otimes Y,Z}"] & (X \otimes Y) \otimes Z
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \otimes 1 \arrow[rr, "\mathcal{Q}_{X,1}"] \arrow[dr, "\eta^r"] & & X \otimes 1 \arrow[dl, "\eta^r"] \\
& X &
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes X \arrow[rr, "\mathcal{Q}_{1,X}"] \arrow[dr, "\eta^l"] & & 1 \otimes X \arrow[dl, "\eta^l"] \\
& X &
\end{tikzcd}$$

In the case of $\mathfrak{C} = \mathfrak{C}'$ and $\Phi = \mathrm{id}$ we say that $\mathcal{Q}$ is a *quantization of the category* $\mathfrak{C}$. The above diagram for quantizations $\mathcal{Q}_{X,Y} : X \otimes Y \longrightarrow X \otimes Y$ of the category $\mathfrak{C}$ takes now the following form

$$\begin{CD}
X \otimes Y \otimes Z @>{id_X \otimes \mathcal{Q}_{Y,Z}}>> X \otimes X \otimes Z \\
@V{\mathcal{Q}_{X,Y} \otimes id_Z}VV @VV{\mathcal{Q}_{X,Y\otimes Z}}V \\
X \otimes Y \otimes Z @>{\mathcal{Q}_{X\otimes Y,Z}}>> X \otimes Y \otimes Z
\end{CD}$$

A *monoidal functor* is a functor $\Phi : \mathfrak{C} \longrightarrow \mathfrak{C}'$ between $\mathbf{k}$-linear monoidal categories which preserves units and admits regular quantizations. A functor is called *lax monoidal* (see [15]) if it admits quantizations which are not necessarily regular.

Theorem 2.9 *[6] Let σ be a braiding and $\mathcal{Q}$ be a regular quantization of a monoidal category $\mathfrak{C}$. Define $\sigma^q_{X,Y} : X \otimes Y \to Y \otimes X$ from the following commutative diagram*

$$\begin{CD}
X \otimes Y @>{\sigma^q_{X,Y}}>> Y \otimes X \\
@V{\mathcal{Q}_{X,Y}}VV @VV{\mathcal{Q}_{Y,X}}V \\
X \otimes Y @>{\sigma_{X,Y}}>> Y \otimes X
\end{CD}$$

Then σ^q is a braiding in $\mathfrak{C}$.

Proof. σ^q is a natural isomorphism by the construction. Therefore we need to prove commutativity of hexagon diagrams. Consider, for example, the commutativity of the first one. In the following diagram:

$$
\begin{tikzcd}
 & & (X\otimes Y)\otimes Z \arrow[r, "\sigma^q"] \arrow[d, "\mathcal{Q}\otimes id"] & Z\otimes(X\otimes Y) \arrow[d, "id\otimes\mathcal{Q}"] \arrow[ddrr, "\alpha"] & & \\
 & & (X\otimes Y)\otimes Z \arrow[r, "\sigma^q"] \arrow[d, "\mathcal{Q}"] & Z\otimes(X\otimes Y) \arrow[d, "\mathcal{Q}"] & & \\
X\otimes(Y\otimes Z) \arrow[uurr, "\alpha"] \arrow[dr, "\mathcal{Q}(id\otimes\mathcal{Q})"] \arrow[dd, equal] & & (X\otimes Y)\otimes Z \arrow[r, "\sigma"] & Z\otimes(X\otimes Y) \arrow[dr, "\alpha"] & & (Z\otimes X)\otimes Y \arrow[dl, "\mathcal{Q}(\mathcal{Q}\otimes id)"'] \arrow[dd, equal] \\
 & X\otimes(Y\otimes Z) \arrow[ur, "\alpha"] \arrow[dr, "id\otimes\sigma"] & & & (Z\otimes X)\otimes Y & \\
X\otimes(Y\otimes Z) \arrow[r, "id\otimes\mathcal{Q}"] \arrow[ddrr, "id\otimes\sigma^q"] & X\otimes(Y\otimes Z) \arrow[u, "\mathcal{Q}"] \arrow[dr, "id\otimes\sigma"] & X\otimes(Z\otimes Y) \arrow[r, "\alpha"] & (X\otimes Z)\otimes Y \arrow[ur, "\sigma\otimes id"] & (Z\otimes X)\otimes Y \arrow[u, "\mathcal{Q}"] & (Z\otimes X)\otimes Y \arrow[l, "\mathcal{Q}\otimes id"'] \\
 & & X\otimes(Z\otimes Y) \arrow[u, "\mathcal{Q}"] & (X\otimes Z)\otimes Y \arrow[u, "\mathcal{Q}"'] \arrow[ur, "\sigma\otimes id"] & & \\
 & & X\otimes(Z\otimes Y) \arrow[r, "\alpha"] \arrow[u, "id\otimes\mathcal{Q}"] & (X\otimes Z)\otimes Y \arrow[u, "\mathcal{Q}\otimes id"'] \arrow[uurr, "\sigma^q\otimes id"] & &
\end{tikzcd}
$$

the interior hexagon commutes by the definition of braidings and squares commute because of naturality or definition of quantizations. Therefore, the exterior hexagon commutes and σ^q is a braiding. ■

Example 2.10 (Main) *Quantizations in H-$\mathcal{M}$od are determined by* **quantizers** *([10], [5]), that is elements $q \in H\otimes H$ such that the following conditions*

$$(\Delta\otimes \mathrm{id}_H)(q)\cdot(q\otimes 1) = (\mathrm{id}_H\otimes\Delta)(q)\cdot(1\otimes q)\,,$$

$$(\varepsilon \otimes \mathrm{id}_H)(q) = (\mathrm{id}_H \otimes \varepsilon)(q) = 1,$$
$$q\Delta(h) = \Delta(h)\, q$$

hold for all $h \in H$.

Moreover, the action $\mathcal{Q}_{X,Y} : X \otimes Y \to Y \otimes X$ is given by the formula

$$\mathcal{Q}_{X,Y}(x \otimes y) = q \cdot (x \otimes y)\,.$$

Remark 2.11 *1. Braiding elements are quantizers [10].*

2. If q is a quantizer and σ is a braiding then

$$\sigma_q = q^{-1}\sigma\tau(q)$$

is a braiding too.

3. Calculations of quantizers modulo natural isomorphisms is equivalent to calculation of the 2nd non-linear cohomology group *of H [10].*

2.4 Internal Homomorphisms

Let $X, Y \in \mathcal{O}b(\mathfrak{C})$ be a pair of objects of $\mathfrak{C}$. Consider a functor $\mathbf{M}_{X,Y} : \mathfrak{C}^{\mathrm{op}} \to \mathbf{k}{-}\mathcal{M}od$ where

$$\mathbf{M}_{X,Y} : Z \longmapsto \mathcal{M}or(Z \otimes X, Y)$$

and $\mathbf{M}_{X,Y}(\phi) : \mathcal{M}or(Z' \otimes X, Y) \to \mathcal{M}or(Z \otimes X, Y)$ is induced by $\phi : Z \to Z'$, that is $\mathbf{M}_{X,Y}(\phi)(\psi) = \psi \circ (\phi \otimes \mathrm{id}_X)$.

A category $\mathfrak{C}$ is called *closed* [11] if $\mathbf{M}_{X,Y}$ is a representable functor for all $X, Y \in \mathcal{O}b(\mathfrak{C})$.

This means that one has

1. The representing objects $\mathrm{Hom}(X,Y) \in \mathcal{O}b(\mathfrak{C})$ called *internal homomorphisms,* and

2. Morphisms
$$\mathrm{ev}_{X,Y} : \mathrm{Hom}(X,Y) \otimes X \longrightarrow Y, \tag{1}$$
called *evaluation maps,*

such that any morphism $\phi : Z \otimes X \to Y$ can be represented by the composition

$$\begin{tikzcd} Z \otimes X \arrow[d, "\phi' \otimes id_X"'] \arrow[dr, "\phi"] & \\ Hom(X,Y) \otimes X \arrow[r, "ev_{X,Y}"] & Y \end{tikzcd}$$

for a unique morphism

$$\phi' : Z \longrightarrow \mathrm{Hom}\,(X,Y)\,.$$

In other words one has an isomorphism

$$\mathcal{M}or(Z, \mathrm{Hom}\,(X,Y)) \overset{\sim}{\longrightarrow} \mathcal{M}or(Z \otimes X, Y)$$

where

$$\phi' \longmapsto \phi = \mathrm{ev}_{X,Y} \circ (\phi' \otimes \mathrm{id}_X)\,.$$

Below we list the main properties of the internal homomorphisms (see, for example, [2], [14]):

- Any morphism $\phi : Y \to Y'$ determines morphism $\phi_* : \mathrm{Hom}\,(X,Y) \to \mathrm{Hom}\,(X,Y')$ through the composition

$$\mathrm{Hom}\,(X,Y) \otimes X \overset{\mathrm{ev}_{X,Y}}{\longrightarrow} Y \overset{\phi}{\longrightarrow} Y'.$$

- Any morphism $\phi : X' \to X$ determines morphism $\phi^* : \mathrm{Hom}\,(X,Y) \to \mathrm{Hom}\,(X',Y)$ through the composition

$$\mathrm{Hom}\,(X,Y) \otimes X' \overset{\mathrm{id}\otimes\phi}{\longrightarrow} \mathrm{Hom}\,(X,Y) \otimes X \overset{\mathrm{ev}_{X,Y}}{\longrightarrow} Y.$$

- The standard properties

$$(\phi \circ \psi)_* = \phi_* \circ \psi_*,\ (\mathrm{id})_* = \mathrm{id},$$

  and

$$(\phi \circ \psi)^* = \psi^* \circ \phi^*,\ (\mathrm{id})^* = \mathrm{id}$$

  hold.

- The composition of morphisms

$$\mathrm{Hom}\,(Y,Z)\otimes \mathrm{Hom}\,(X,Y)\otimes X \overset{\mathrm{id}\,\otimes\, \mathrm{ev}_{X,Y}}{\longrightarrow} \mathrm{Hom}\,(Y,Z)\otimes Y \overset{\mathrm{ev}_{Y,Z}}{\longrightarrow} Z$$

  determines a *composition of internal homomorphisms*

$$\begin{aligned} \mathrm{Hom}\,(Y,Z)\otimes \mathrm{Hom}\,(X,Y) &\longrightarrow \mathrm{Hom}\,(X,Z)\,, \\ f\otimes g &\longmapsto f\diamond g \end{aligned}$$

- There is a natural isomorphism

$$\mathrm{Hom}\,(Z,\mathrm{Hom}\,(X,Y)) \longrightarrow \mathrm{Hom}\,(Z\otimes X,Y)$$

  given by the following chain of isomorphisms

$$\begin{aligned} &\mathcal{M}or\,(T,\mathrm{Hom}\,(Z,\mathrm{Hom}\,(X,Y))) \longrightarrow \mathcal{M}or\,(T\otimes Z,\mathrm{Hom}\,(X,Y)) \to \\ &\qquad \to\ \mathcal{M}or\,(T\otimes Z\otimes X,Y) \leftarrow \mathcal{M}or\,(T,\mathrm{Hom}\,(Z\otimes X,Y)) \end{aligned}$$

- A unity object $\mathbf{1}$ yields an isomorphism

$$l_X : X \longrightarrow \mathrm{Hom}\,(\mathbf{1},X) \tag{2}$$

  corresponding to $\eta : X\otimes\mathbf{1} \to X$ by the isomorphism $\mathcal{M}or\,(X,\mathrm{Hom}\,(\mathbf{1},X)) \to \mathcal{M}or\,(X\otimes\mathbf{1},X)$.

- Using isomorphism (2) we shall consider the evaluation morphism (1) as a composition:

$$\begin{tikzcd} Hom(X,Y)\otimes X \arrow[r, "ev_{X,Y}"] \arrow[d, "id\otimes l_X"'] & Y \arrow[d, "l_Y"'] \\ Hom(X,Y)\otimes Hom(1,X) \arrow[r] & Hom(1,Y) \end{tikzcd}$$

- By using the following morphisms

$$\begin{array}{c} \mathrm{Hom}\,(X_1,Y_1)\otimes \mathrm{Hom}\,(X_2,Y_2)\otimes X_1\otimes X_2 \\ \Big\downarrow \mathrm{id}\otimes\sigma\otimes\mathrm{id} \\ \mathrm{Hom}\,(X_1,Y_1)\otimes X_1\otimes \mathrm{Hom}\,(X_2,Y_2)\otimes X_2 \\ \Big\downarrow \mathrm{ev}_{X_1,Y_1}\otimes \mathrm{ev}_{X_2,Y_2} \\ Y_1\otimes Y_2 \end{array}$$

  we get a morphism

$$\mathrm{Hom}\,(X_1,Y_1)\otimes \mathrm{Hom}\,(X_2,Y_2)\longrightarrow \mathrm{Hom}\,(X_1\otimes X_2, Y_1\otimes Y_2)\,.$$

Relations between tensor products and internal homomorphisms become more visible if we apply the following construction.

Any $P\in \mathcal{O}b\,(\mathfrak{C})$ determines a covariant functor

$$h_P : \mathfrak{C}\to\mathfrak{C}$$

where $h_P\,(X) = \mathrm{Hom}\,(P,X)$ and $h_P\,(\phi) = \phi_*$ for $X\in\mathcal{O}b\,(\mathfrak{C})$ and $\phi\in\mathcal{M}or\,(X,Y)$.

Theorem 2.12 1. *The associativity constraint establishes natural isomorphism between composition $h_P\circ h_Q$ and $h_{P\otimes Q}$ for all $P,Q\in\mathcal{O}\mathrm{b}\,(\mathfrak{C})$. Thus, $h_{P\otimes Q} = h_P\circ h_Q$ in strict categories.*

2. *Morphisms $\phi : P\to Q$ induce natural isomorphisms of functors $\phi^* : h_Q\to h_P$.*

Example 2.13 (Main) $\mathrm{Hom}\,(X,Y)$ *is the module of* $\mathbf{k}$*-morphisms* $\mathrm{Hom}_{\mathbf{k}}\,(X,Y)$ *equipped with the following H-module structure:*

$$h\,(f)\,(x) = \sum h_{(1)} f\left(S\left(h_{(2)}\right)x\right)$$

for all $h\in H,\ x\in X,\ f\in\mathrm{Hom}_{\mathbf{k}}\,(X,Y)$.

2.5 What do we quantize?

Let $\mathcal{Q}$ be a quantization of the category $\mathfrak{C}$. As we see any braiding σ generates a new braiding σ^q of $\mathfrak{C}$. We call σ^q a *quantization of* σ. In this section we shall discuss quantizations of other algebraic objects. We start with

Algebras

Let $\mu : A \otimes A \to A$ be an algebra in $\mathfrak{C}$. Define a *quantization* of the algebra as the same object A equipped with the following multiplication

$$\mu_q : A \otimes A \xrightarrow{\mathcal{Q}_{A,A}} A \otimes A \xrightarrow{\mu} A.$$

Theorem 2.14 *(A, μ_q) is an algebra in $\mathfrak{C}$.*
Proof. Let us prove the associativity law for μ_q. Since $\mathcal{Q}$ is a natural transformation we have the following commutative diagram

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "id_A \otimes \mu"] \arrow[d, "\mathcal{Q}_{A,A\otimes A}"] & A \otimes A \arrow[d, "\mathcal{Q}_{A,A}"] \\
A \otimes A \otimes A \arrow[r, "id_A \otimes \mu"] & A \otimes A
\end{tikzcd}$$

Therefore

$$\begin{aligned}
\mu_q \circ (\mathrm{id}_A \otimes \mu_q) &= \mu \circ \mathcal{Q}_{A,A} \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_A \otimes \mathcal{Q}_{A,A}) \\
&= \mu \circ (\mathrm{id}_A \otimes \mu) \circ \mathcal{Q}_{A,A\otimes A} \circ (\mathrm{id}_A \otimes \mathcal{Q}_{A,A}),
\end{aligned}$$

and

$$\begin{aligned}
\mu_q \circ (\mu_q \otimes \mathrm{id}_A) &= \mu \circ \mathcal{Q}_{A,A} \circ (\mu \otimes \mathrm{id}_A) \circ (\mathcal{Q}_{A,A} \otimes \mathrm{id}_A) \\
&= \mu \circ (\mu \otimes \mathrm{id}_A) \circ \mathcal{Q}_{A\otimes A,A} \circ (\mathcal{Q}_{A,A} \otimes \mathrm{id}_A).
\end{aligned}$$

But

$$\mathcal{Q}_{A\otimes A,A} \circ (\mathcal{Q}_{A,A} \otimes \mathrm{id}_A) = \mathcal{Q}_{A,A\otimes A} \circ (\mathrm{id}_A \otimes \mathcal{Q}_{A,A})$$

is the coherence condition for quantizations. ■

Modules

Let (A, μ) be an algebra in $\mathfrak{C}$ and (X, μ^l) be a left (right, bi-) module in $\mathfrak{C}$. A *quantization of the module* is the same object X equipped with new multiplication

$$\mu_q^l : A \otimes X \xrightarrow{\mathcal{Q}_{A,X}} A \otimes X \xrightarrow{\mu^l} X.$$

As above we get the following
Theorem 2.15 *A quantization (X, μ_q^l) is a left (A, μ_q) - module in $\mathfrak{C}$.*

Coalgebras

Let A be a coalgebra in $\mathfrak{C}$ and let $\Delta : A \to A \otimes A$ be the diagonal. As above we define a *quantization of the coalgebra* to be the same object A equipped with a new diagonal

$$\Delta_q : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathcal{Q}^{-1}_{A,A}} A \otimes A.$$

Here we assume that $\mathcal{Q}$ is a regular quantization.

Theorem 2.16 *A quantization (A, Δ_q) is a coalgebra in $\mathfrak{C}$.*

Bialgebras

Let (A, μ) be an algebra in $\mathfrak{C}$. We consider the tensor square $A^{\otimes 2} = A \otimes A$ as an algebra with respect to multiplication μ^2_σ given by the following commutative diagram

$$\begin{tikzcd}
(A \otimes A) \otimes (A \otimes A) \arrow[r, "{\mu^2}_\sigma"] \arrow[d, "id_A \otimes \sigma_{A,A} \otimes id_A"] & A \otimes A \arrow[d, equal] \\
(A \otimes A) \otimes (A \otimes A) \arrow[r, "\mu \otimes \mu"] & A \otimes A.
\end{tikzcd}$$

Lemma 2.17 *The morphism μ^2_σ determines an associative algebra structure on $A^{\otimes 2}$.*

Definition 2.18 *A σ-bialgebra (A, μ, Δ) in a category $\mathfrak{C}$ is an algebra (A, μ) and coalgebra (A, Δ) in the category such that the diagonal $\Delta : (A, \mu) \to \left(A^{\otimes 2}, \mu^2_\sigma\right)$ is an algebra morphism.*

Theorem 2.19 *Let (A, μ, Δ) be a σ-algebra in category $\mathfrak{C}$. Then (A, μ_q, Δ_q) is a σ_q-bialgebra in the category.*

Internal homomorphisms

Denote by

$$\begin{aligned}
&\mu^c : \mathrm{Hom}\,(Y, Z) \otimes \mathrm{Hom}\,(X, Y) \longrightarrow \mathrm{Hom}\,(X, Z)\,, \\
&\mu^c : f \otimes g \mapsto f * g
\end{aligned}$$

the composition of internal homomorphisms.

We shall consider the class of all internal homomorphisms as an algebra equipped with partially determined operation $*$ and quantization of internal homomorphisms as a quantization of this algebra.

Namely, we define a *quantization* of internal homomorphisms to be a natural isomorphism

$$\mathcal{Q}_h : \mathrm{Hom}\,(X,Y) \longrightarrow \mathrm{Hom}\,(X,Y)$$

such that

$$\mathcal{Q}_h\,(f) * x = f \underset{\mathcal{Q}}{*} x$$

for all $f \in \mathrm{Hom}\,(X,Y)\,,\ \ x \in X = \mathrm{Hom}\,(\mathbf{1}, X)\,.$

Here

$$f \underset{\mathcal{Q}}{*} g = \mu^c\,(\mathcal{Q}_{H_1,H_2}\,(f \otimes g))$$

where $H_1 = \mathrm{Hom}\,(X,Y)\,,\ H_2 = \mathrm{Hom}\,(Y,Z)$

Theorem 2.20 *[8]*

1. $\mathcal{Q}_h\,(x) = x,$
2. $\mathcal{Q}_h\,(f * g) = \mathcal{Q}_h\,(f) * \mathcal{Q}_h\,(g)$

for all $x \in X = \mathrm{Hom}\,(\mathbf{1}, X)\,,\ \ f \in \mathrm{Hom}\,(X,Y)\,,\ g \in \mathrm{Hom}\,(Y,Z)$

2.6 σ-Commutative algebras

Let A be a $\mathfrak{C}$-algebra, that is an algebra in the category $\mathfrak{C}$ with multiplication $\mu_A : A \otimes A \to A$ and unit $e : \mathbf{1} \to A$.

We say that A is a *σ-commutative algebra* if the following diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\sigma_{A,A}"] \arrow[dr, "\mu_A"'] & & A \otimes A \arrow[dl, "\mu_A"] \\ & A & \end{tikzcd}$$

commutes.

Example 2.21 (Main) *An H-algebra A is σ-commutative if and only if*

$$a \cdot b = \sum \sigma'\,(b) \cdot \sigma''\,(a)$$

for all $a, b \in H$.

Example 2.22 (G-graded) *An algebra A is a G-graded σ-commutative algebra iff:*

$$A = \coprod_{g \in G} A_g, \quad A_g \cdot A_h \subset A_{gh}.$$

and

$$a_g a_h = s(g,h)\, a_h a_g$$

for all $a_g \in A_g$, $a_h \in A_h$, $g, h \in G$.

Example 2.23 (Color) *We consider a special case of color commutative algebras, namely the group algebra $\mathbf{k}[G]$. For more examples see [9]. The group algebra is obviously G-graded. Introduce the following G-action*

$$g(\delta_h) = \delta_{ghg^{-1}}.$$

This algebra is σ-commutative with respect to braiding with $s = 1$:

$$\delta_g \delta_h = g(\delta_h)\, \delta_g.$$

2.7 σ-Symmetric bimodules

Let A be a σ-commutative algebra and let X be a A-A-bimodule in $\mathfrak{C}$. Denote by

$$\mu^l : A \otimes X \to X \qquad \text{and} \qquad \mu^r : X \otimes A \to X$$

left and right multiplications respectively.

We say that X is a *σ-symmetric bimodule* if the following diagrams

$$\begin{tikzcd} X \otimes A \arrow[rr, "\sigma_{X,A}"] \arrow[dr, "\mu^r"'] & & A \otimes X \arrow[dl, "\mu^l"] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} A \otimes X \arrow[rr, "\sigma_{A,X}"] \arrow[dr, "\mu^l"'] & & X \otimes A \arrow[dl, "\mu^r"] \\ & X & \end{tikzcd}$$

commute.

Definition 2.24 *Let X and Y be σ-symmetric A-A-bimodules. Define, as usual, their tensor product $X \underset{A}{\otimes} Y$ as the cokernel*

$$\mathbf{coker}\left(X \otimes A \otimes Y \underset{\mu^r \otimes \mathrm{id}_Y}{\overset{\mathrm{id}_X \otimes \mu^l}{\rightrightarrows}} X \otimes Y \right) =: X \underset{A}{\otimes} Y.$$

Lemma 2.25 *The following diagrams*

$$\begin{tikzcd} X \otimes A \otimes Y \arrow[r, "id_X \otimes \mu^l"] \arrow[d, "\mathfrak{Y}_{\mathfrak{X},\mathfrak{A},\mathfrak{Y}}"] & X \otimes Y \arrow[d, "\sigma_{X,Y}"] \\ Y \otimes A \otimes X \arrow[r, "\mu^r \otimes id_X"] & Y \otimes X \end{tikzcd}$$

$$\begin{tikzcd} X \otimes A \otimes Y \arrow[r, "\mu^r \otimes id_X"] \arrow[d, "\mathfrak{Y}_{\mathfrak{X},\mathfrak{A},\mathfrak{Y}}"] & X \otimes Y \arrow[d, "\sigma_{X,Y}"] \\ Y \otimes A \otimes X \arrow[r, "id_X \otimes \mu^l"] & Y \otimes X \end{tikzcd}$$

commute.

Proof. Let us prove, for example, the commutativity of the first diagram. Using the naturality of braiding we have the following commutative diagram

$$\begin{tikzcd} X \otimes (A \otimes Y) \arrow[r, "id_X \otimes \mu^l"] \arrow[d, "\sigma_{X,A\otimes Y}"] & X \otimes Y \arrow[d, "\sigma_{X,Y}"] \\ (A \otimes Y) \otimes X \arrow[r, "\mu^l \otimes id_X"] & Y \otimes X \end{tikzcd}$$

Note that $\mu^l = \mu^r \circ \sigma_{A,Y}$ by the definition of a σ-symmetric module and the relation $\mathfrak{Y}_{X,A,Y} = (\sigma_{A,Y} \otimes \mathrm{id}_X) \circ \sigma_{X,A\otimes Y}$. ■

Denote by $\overline{\sigma}_{X,Y} : X \underset{A}{\otimes} Y \to Y \underset{A}{\otimes} X$ morphism induced by $\sigma_{X,Y}$.

Lemma 2.26 *1. Let Y be a σ-symmetric A-A-bimodule. Then $\overline{\sigma}_{X,Y}$ is a left A-module morphism.*

2. *Let X be a σ-symmetric A-A-bimodule. Then $\overline{\sigma}_{X,Y}$ is a right A-module morphism.*

3. *Let X and Y be σ-symmetric A-A-bimodules. Then $\overline{\sigma}_{X,Y}$ is an A-A-bimodule morphism.*

Proof. We prove the first statement. The following diagram commutes due to the naturality of σ :

$$\begin{tikzcd} A \otimes X \otimes Y \arrow[r, "\sigma_{A\otimes X,Y}"] \arrow[d, "\mu^l \otimes id_Y"] & Y \otimes A \otimes X \arrow[d, "id_Y \otimes \mu^l"] \\ X \otimes Y \arrow[r, "\sigma_{X,Y}"] & Y \otimes X \end{tikzcd}$$

On the other side $\mathrm{id}_Y \otimes \mu^l = \mu^r \otimes \mathrm{id}_X$ on $X \underset{A}{\otimes} Y$ and

$$\mu^r \otimes \mathrm{id}_X = \left(\mu^l \otimes \mathrm{id}_X\right) \circ \left(\sigma_{Y,A} \otimes \mathrm{id}_X\right)$$

because Y is σ-symmetric.

Moreover, $(\sigma_{Y,A} \otimes \mathrm{id}_X) \circ \sigma_{A\otimes X,Y} = (\mathrm{id_A} \otimes \sigma_{X,Y})$, and therefore we get the following commutative diagram

$$\begin{tikzcd} A \otimes X \otimes_A Y \arrow[r, "id_A \otimes \overline{\sigma}_{X,Y}"] \arrow[d, "\mu^l"] & A \otimes Y \otimes_A X \arrow[d, "\mu^l"] \\ X \otimes_A Y \arrow[r, "\overline{\sigma}_{X,Y}"] & Y \otimes_A X \end{tikzcd}$$

and the statement. ■

Theorem 2.27 *Let $\mathfrak{SYM}(A) \subset \mathfrak{C}$ be a category of σ-symmetric bimodules over σ-commutative algebra $A \in \mathcal{O}\mathrm{b}(\mathfrak{C})$. Then*

1. *$\mathfrak{SYM}(A)$ is a monoidal category with respect to $\otimes_A$. Morphisms in $\mathfrak{SYM}(A)$ are morphisms of $\mathfrak{C}$ which are A-A-homomorphisms.*

2. *The braiding σ induces a braiding in $\mathfrak{SYM}(A)$.*

Proof. Due to the above lemma it is enough to show that the tensor product of two σ-symmetric modules is σ-symmetric, too. To see this we compare the bottom and the top arrows of diagrams

$$\begin{tikzcd}
X \otimes Y \otimes A \arrow[r, "id_X \otimes \mu^r"] \arrow[d, "id_X \otimes \sigma_{Y,A}"] & X \otimes Y \arrow[d, equal] \\
X \otimes A \otimes Y \arrow[r, "id_X \otimes \mu^l"] & X \otimes Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes A \otimes Y \arrow[r, "\mu^r \otimes id_Y"] \arrow[d, "\sigma_{Y,A} \otimes id_X"] & X \otimes Y \arrow[d, equal] \\
X \otimes A \otimes Y \arrow[r, "\mu^l \otimes id_Y"] & X \otimes Y
\end{tikzcd}$$

■

2.8 A-modules

Let (X, μ^l) be a left A-module in $\mathfrak{C}$ with multiplication $\mu^l : A \otimes X \to X$. Define a right multiplication μ^r_σ from the diagram

$$\begin{tikzcd}
X \otimes A \arrow[rr, "\sigma_{X,A}"] \arrow[dr, "{\mu_\sigma}^r"] & & A \otimes X \arrow[dl, "\mu^l"] \\
& X & X
\end{tikzcd}$$

Proposition 2.28 *(X, μ^l, μ^r_σ) is an A-A-bimodule.*

Proof. Look at the diagram

$$\begin{tikzcd}
X \otimes A \otimes A \arrow[r, "id_X \otimes \mu_A"] \arrow[d, "id_X \otimes \sigma_{A,A}"] & X \otimes A \arrow[r, "{\mu_\sigma}^r"] \arrow[d, equal] & X \arrow[d, equal] \\
X \otimes A \otimes A \arrow[r, "id_X \otimes \mu_A"] \arrow[d, "\sigma_{X,A \otimes A}"] & X \otimes A \arrow[r, "{\mu_\sigma}^r"] \arrow[d, "\sigma_{X,A}"] & X \arrow[d, equal] \\
A \otimes A \otimes X \arrow[r, "\mu_A \otimes id_X"] & A \otimes X \arrow[r, "\mu^l"] & X
\end{tikzcd}$$

The first square in the diagram commutes because of σ-commutativity of A, and the second one commutes by naturality of σ.

Therefore we get

$$\mu^r_\sigma \circ (\mathrm{id}_X \otimes \mu_A) = \mu^l \circ (\mu_A \otimes \mathrm{id}_X) \circ \mathfrak{Y}_{X,A,A}.$$

On the other side, from the diagram

$$\begin{tikzcd}
X \otimes A \otimes A \arrow[r, "{\mu^r}_s \otimes id_A"] \arrow[d, "\sigma_{X,A} \otimes id_A"] & X \otimes A \arrow[r, "{\mu^r}_s"] \arrow[d, equal] & X \arrow[d, equal] \\
A \otimes X \otimes A \arrow[r, "\mu^l \otimes id_A"] \arrow[d, "\sigma_{A \otimes X,A}"] & X \otimes A \arrow[r, "{\mu^r}_s"] \arrow[d, "\sigma_{X,A}"] & X \arrow[d, equal] \\
A \otimes A \otimes X \arrow[r, "id_A \otimes \mu^l"] & A \otimes X \arrow[r, "\mu^l"] & X
\end{tikzcd}$$

we get

$$\mu^r_\sigma \circ (\mu^r_\sigma \otimes \mathrm{id}_A) = \mu^l \circ \left(\mathrm{id}_A \otimes \mu^l\right) \circ \mathfrak{Y}_{X,A,A}.$$

Therefore (X, μ^r_σ) is a right A-module.

To prove that μ^l and μ^r_σ commute we should compare the following two morphisms:

$$F = \mu^r \circ \left(\mu^l \otimes \mathrm{id}_A\right)$$
$$G = (\mathrm{id}_A \otimes \mu^r) \circ \mu^l.$$

Due to naturality of σ we have the following commutative diagram

$$\begin{tikzcd} A \otimes X \otimes A \arrow[r, "\sigma_{A\otimes X,A}"] \arrow[d, "\mu^l \otimes id_A"'] & A \otimes A \otimes X \arrow[d, "id_A \otimes \mu^l"] \\ X \otimes A \arrow[r, "\sigma_{X,A}"] & A \otimes X \end{tikzcd}$$

and therefore

$$F = \mu^l \circ \left(\mathrm{id}_A \otimes \mu^l\right) \circ \sigma_{A\otimes X,A} = \mu^l \circ (\mu_A \otimes \mathrm{id}_X) \circ \sigma_{A\otimes X,A},$$
$$G = \mu^l \circ \left(\mathrm{id}_A \otimes \mu^l\right) \circ (\mathrm{id}_A \otimes \sigma_{X,A}) = \mu^l \circ (\mu_A \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \sigma_{X,A}) .$$

But $\sigma_{A\otimes X,A} = (\sigma_{A,A} \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \sigma_{X,A})$.

Hence the statement follows from the σ-commutativity of A. ■

Now we transfer structures from the A-A-bimodules to A-modules.

Definition 2.29 *1. We call a left A-module $\left(X, \mu^l\right)$* σ-symmetric *if A-A-bimodule $\left(X, \mu^l, \mu^r_\sigma\right)$ is σ-symmetric.*

2. By tensor product $X \underset{A}{\otimes} Y$ of A-modules we mean the tensor $(\underset{A}{\otimes})$ product of corresponding A-A-bimodules equipped with the structure of left A-module.

Summing over the above statements we get the following

Theorem 2.30 *Let X and Y are left modules over σ-commutative algebra. Then*

1. *Braiding $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ induced morphism $\overline{\sigma}_{X,Y} : X \underset{A}{\otimes} Y \to Y \underset{A}{\otimes} X$.*
2. *$\overline{\sigma}_{X,Y}$ is a morphism of left A-modules if Y is a σ-symmetric A-module.*

Corollary 2.31 *$\overline{\sigma}$ determines a braiding in the category of σ-symmetric A-modules.*

3 Products

3.1 K- and G-theory

Many of the usual notions of the theory of modules can be easily transferred to the category of A-modules over a $\mathfrak{C}$-algebra A. It should be clear what is meant when we say "left/right projective, left/right flat A-modules, left/right Noetherian algebras" etc. We need, however, some additional restrictions on the category $\mathfrak{C}$:

1. $\mathfrak{C}$ is equivalent to a small category;
2. $\mathfrak{C}$ has enough projectives;
3. All projectives in $\mathfrak{C}$ are flat;
4. Tensor product of projectives in $\mathfrak{C}$ is always projective;
5. The unit object $\mathbf{1}$ is projective.

Our main example will be $\mathfrak{C} = H$-**mod**, where the latter is the category of finitely generated left H-modules over a quasi-triangular Hopf algebra H. It is a slight modification of the main example 2.1. The category satisfies the conditions above whenever H is projective and finitely generated as a k-module.

We include now some more or less evident facts in the following

Theorem 3.1 *a) A left A-module is projective if and only if it is a direct summand of a module of the form $A \otimes P$ where P is a projective in $\mathfrak{C}$;*

b) Left projective A-modules are left flat. If A is σ-commutative they are right flat, too;

c) If A is σ-commutative and P and Q are left A-projective then $P \otimes_A Q$ is left A-projective;

d) A is a projective left A-module.

Proof. a) For a $\mathfrak{C}$-projective P the functor $\mathcal{M}or_A(A \otimes P, \_)$ is isomorphic to the functor $\mathcal{M}or(P, \_)$ which is exact. Therefore, $A \otimes P$ is left A-projective. Given a projective A-module M, there exists a $\mathfrak{C}$-epimorphism $P \twoheadrightarrow M$ for some $\mathfrak{C}$-projective P. The induced A-homomorphism $A \otimes P \twoheadrightarrow M$ is a splitting epimorphism, and M is a direct summand of $A \otimes P$.

b) and c) can be proved using a). It is sufficient to check b) and c) only for modules of the form $A \otimes P$, which is easy.

d) $A \approx A \otimes \mathbf{1}$. ■

Now we are prepared to define K_*- and G_*-functors.

Definition 3.2 *Given a $\mathfrak{C}$-algebra A let us define*

$$K_*(A) := K_*(\mathbf{P}(A)) = \boldsymbol{\pi}_{*+1}(BQ\mathbf{P}(A)) = \boldsymbol{\pi}_*(\Omega BQ\mathbf{P}(A)),$$

$$K_*^{sym}(A) := K_*(\mathbf{P}^{sym}(A)) = \boldsymbol{\pi}_{*+1}(BQ\mathbf{P}^{sym}(A)) = \boldsymbol{\pi}_*(\Omega BQ\mathbf{P}^{sym}(A)),$$

$$G_*(A) := K_*(A\text{-}\mathbf{mod}) = \boldsymbol{\pi}_{*+1}(BQ\,(A\text{-}\mathbf{mod})\,) = \boldsymbol{\pi}_*(\Omega BQ\,(A\text{-}\mathbf{mod})\,)$$

$$G_*^{sym}(A) := K_*\left((A\text{-}\mathbf{mod})^{sym}\right) = \boldsymbol{\pi}_{*+1}(BQ\,(A\text{-}\mathbf{mod})^{sym}\,) = \boldsymbol{\pi}_*(\Omega BQ\,(A\text{-}\mathbf{mod})^{sym}\,)$$

in the sense of Daniel Quillen (see, for instance, [13] or [16]) where $\mathbf{P}(A)$ *is the category of projective left* A*-modules,* $\mathbf{P}^{sym}(A)$ *is the category of projective symmetric left* A*-modules,* A-**mod** *is the category of left* A*-modules, and* $(A\text{-}\mathbf{mod})^{sym}$ *is the category of symmetric left* A*-modules.*

Remark 3.3 *In our main example* $\mathfrak{C} = H$-**mod** *is the category of* **finitely generated** *left* H*-modules, and therefore both* $\mathfrak{C}$ *and* A-**mod** *can be considered as* **small** *categories, which implies that the definitions are correct.*

3.2 Graded rings

Definition 3.4 *Given a graded ring* $R = \coprod_{n\geq 0} R_n$*, the subring*

$$Z(R) = \coprod_{n\geq 0} Z_n$$

where

$$Z_n = \{r \in R_n | \forall m \forall s \in R_m (rs = (-1)^{mn} sr)\}$$

is called the **graded center** *of* R.

The graded ring R *is called* **graded commutative** *if* $Z(R) = R$.

It is easy to see that $Z(R)$ is a graded commutative ring.

Definition 3.5 *Given a graded* R*-*R*-bimodule* P*, the subbimodule*

$$Sym(P) = \coprod_{n\geq 0} S_n$$

where

$$S_n = \{p \in P_n | \forall m \forall r \in R_m (rp = (-1)^{mn} pr)\}$$

is called the **symmetric part** *of* P.

The subring

$$Z(P) = \coprod_{n\geq 0} Z_n$$

where

$$Z_n = \{r \in R_n | \forall m \forall p \in P_m (rp = (-1)^{mn} pr)\}$$

is called the **symmetrizer** *of* P.

Definition 3.6 *A graded R-R-bimodule P is called* **symmetric** *if*

$$P = Sym(P),$$

or, equivalently, if

$$Z(P) = R.$$

Clearly $Sym(Sym(P)) = Sym(P)$ and $Sym(P)$ is a symmetric bimodule. Of course,

$$Z(\text{the ring } R) = Z(\text{the } R\text{-}R\text{-bimodule } R).$$

3.3 Multiplicative structure

Theorem 3.7 *a) Any bi-exact functor*

$$F : \mathbf{A} \times \mathbf{B} \longrightarrow \mathbf{C}$$

induces a pairing

$$\Omega BQ\mathbf{A} \wedge \Omega BQ\mathbf{B} \longrightarrow \Omega BQ\mathbf{C};$$

b) The pairing is homotopy associative in the following sense. Given exact categories $\mathbf{A}_i$, $i = 0, 1, 2$, $\mathbf{A}_{i,i+1}$, $i = 0, 1$, and $\mathbf{A}_{012}$, and given bi-exact functors

$$\mathbf{A}_i \times \mathbf{A}_{i+1} \longrightarrow \mathbf{A}_{i,i+1},$$

$$\mathbf{A}_0 \times \mathbf{A}_{12} \longrightarrow \mathbf{A}_{012},$$

$$\mathbf{A}_{01} \times \mathbf{A}_2 \longrightarrow \mathbf{A}_{012},$$

which are associative, such that the diagram

$$\begin{array}{ccc} \mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2 & \longrightarrow & \mathbf{A}_0 \times \mathbf{A}_{12} \\ \downarrow & & \downarrow \\ \mathbf{A}_{01} \times \mathbf{A}_2 & \longrightarrow & \mathbf{A}_{012} \end{array}$$

commutes up to a functor isomorphism, then the following diagram

$$\begin{array}{ccc} \Omega BQ\mathbf{A}_0 \wedge \Omega BQ\mathbf{A}_1 \wedge \Omega BQ\mathbf{A}_2 & \longrightarrow & \Omega BQ\mathbf{A}_0 \wedge \Omega BQ\mathbf{A}_{12} \\ \downarrow & & \downarrow \\ \Omega BQ\mathbf{A}_{01} \wedge \Omega BQ\mathbf{A}_2 & \longrightarrow & \Omega BQ\mathbf{A}_{012} \end{array}$$

commutes up to a basepoint-preserving homotopy;

c) The pairing is homotopy commutative in the following sense: The diagram

$$\begin{array}{ccc} \Omega BQ\mathbf{A} \wedge \Omega BQ\mathbf{B} & \xrightarrow{\mu} & \Omega BQ\mathbf{C} \\ \downarrow \tau & & =\downarrow \\ \Omega BQ\mathbf{B} \wedge \Omega BQ\mathbf{A} & \xrightarrow{\nu} & \Omega BQ\mathbf{C} \end{array}$$

commutes up to basepoint-preserving homotopy, where τ is a natural twist, and ν is induced by the bi-exact functor

$$(Y, X) \longmapsto F(X, Y).$$

Proof. See [17], [16], and ([3], (2.1-2.4) and Appendix). ■

Theorem 3.8 *Given a σ-commutative algebra A:*

a) There exist associative ring structures on the K-theory:

$$K_*(A) \times K_*(A) \longrightarrow K_*(A),$$
$$K_*^{sym}(A) \times K_*^{sym}(A) \longrightarrow K_*^{sym}(A);$$

b) There exists a structure of a $K_*(A)$-$K_*(A)$-bimodule on $G_*(A)$ (A left Noetherian);

c) The ring $K_*^{sym}(A)$ is graded commutative (definition 3.4);

d) The ring $K_*^{sym}(A)$ is mapped under the natural ring homomorphism

$$K_*^{sym}(A) \longrightarrow K_*(A)$$

into the center $Z(K_*(A))$ in the general case, or into the intersection

$$Z(K_*(A)) \cap Z(G_*(A))$$

of graded center of $K_*(A)$ and the symmetrizer of $G_*(A)$ (definitions 3.4 and 3.5) whenever A is left Noetherian;

e) The graded Abelian group $G_*^{sym}(A)$ is mapped under the natural homomorphism

$$G_*^{sym}(A) \longrightarrow G_*(A)$$

into the symmetric part $Sym(G_*(A))$ (A left Noetherian).

Proof. Consider the bifunctor

$$\otimes_A : A\text{-}\mathbf{mod} \times A\text{-}\mathbf{mod} \longrightarrow A\text{-}\mathbf{mod}$$

(definition 2.24). This bifunctor is associative up to isomorphism in the sense that the two compositions

$$A\text{-}\mathbf{mod} \times A\text{-}\mathbf{mod} \times A\text{-}\mathbf{mod} \overset{\otimes_A \times Id_{A-\mathbf{mod}}}{\longrightarrow} A\text{-}\mathbf{mod} \times A\text{-}\mathbf{mod} \overset{\otimes_A}{\longrightarrow} A\text{-}\mathbf{mod}$$

and

$$A\text{-}\mathbf{mod} \times A\text{-}\mathbf{mod} \times A\text{-}\mathbf{mod} \overset{Id_{A\text{-}\mathbf{Mod}} \times \otimes_A}{\longrightarrow} A\text{-}\mathbf{mod} \times A\text{-}\mathbf{mod} \overset{\otimes_A}{\longrightarrow} A\text{-}\mathbf{mod}$$

are isomorphic functors. Now consider the restrictions of the functors above to the subcategory $\mathbf{P}(A)$. It follows from Theorem 3.7 that there exist pairings

$$\Omega BQ\mathbf{P}(A) \wedge \Omega BQ\mathbf{P}(A) \xrightarrow{\mu} \Omega BQ\mathbf{P}(A),$$

$$\Omega BQ\mathbf{P}(A) \wedge \Omega BQ\left(A\text{-}\mathbf{mod}\right) \xrightarrow{\mu} \Omega BQ\left(A\text{-}\mathbf{mod}\right),$$

$$\Omega BQ\left(A\text{-}\mathbf{mod}\right) \wedge \Omega BQ\mathbf{P}(A) \xrightarrow{\mu} \Omega BQ\left(A\text{-}\mathbf{mod}\right),$$

such that the following diagrams commute up to basepoint-preserving homotopy:

$$\begin{tikzcd}
\Omega BQ\mathbf{P}(A) \wedge \Omega BQ\mathbf{P}(A) \wedge \Omega BQ\mathbf{P}(A) \arrow[r, "\mu \wedge 1"] \arrow[d, "1 \wedge \mu"'] \arrow[dr, phantom, "="] & \Omega BQ\mathbf{P}(A) \wedge \Omega BQ\mathbf{P}(A) \arrow[d, "\mu"] \\
\Omega BQ\mathbf{P}(A) \wedge \Omega BQ\mathbf{P}(A) \arrow[r, "\mu"] & \Omega BQ\mathbf{P}(A)
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega BQ\mathbf{P}(A)\wedge\Omega BQ\mathbf{P}(A)\wedge\Omega BQ\,(A\text{-}\mathbf{mod}) \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] \arrow[dr, phantom, "="] & \Omega BQ\mathbf{P}(A)\wedge\Omega BQ\,(A\text{-}\mathbf{mod}) \arrow[d, "\mu"] \\
\Omega BQ\mathbf{P}(A)\wedge\Omega BQ\,(A\text{-}\mathbf{mod}) \arrow[r, "\mu"] & \Omega BQ\,(A\text{-}\mathbf{mod})
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega BQ\,(A\text{-}\mathbf{mod})\wedge\Omega BQ\mathbf{P}(A)\wedge\Omega BQ\mathbf{P}(A) \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] \arrow[dr, phantom, "="] & \Omega BQ\,(A\text{-}\mathbf{mod})\wedge\Omega BQ\mathbf{P}(A) \arrow[d, "\mu"] \\
\Omega BQ\,(A\text{-}\mathbf{mod})\wedge\Omega BQ\mathbf{P}(A) \arrow[r, "\mu"] & \Omega BQ\,(A\text{-}\mathbf{mod})
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega BQ\mathbf{P}(A)\wedge\Omega BQ\,(A\text{-}\mathbf{mod})\wedge\Omega BQ\mathbf{P}(A) \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] \arrow[dr, phantom, "="] & \Omega BQ\,(A\text{-}\mathbf{mod})\wedge\Omega BQ\mathbf{P}(A) \arrow[d, "\mu"] \\
\Omega BQ\mathbf{P}(A)\wedge\Omega BQ\,(A\text{-}\mathbf{mod}) \arrow[r, "\mu"] & \Omega BQ\,(A\text{-}\mathbf{mod})
\end{tikzcd}$$

This proves statements a) and b) of the theorem.
The following diagram of categories and functors

$$\begin{array}{ccc}
\mathbf{P}^{sym}(A)\times\mathbf{P}(A) & \xrightarrow{\otimes_A} & \mathbf{P}(A) \\
\downarrow\tau & & =\downarrow \\
\mathbf{P}(A)\times\mathbf{P}^{sym}(A) & \xrightarrow{\otimes_A} & \mathbf{P}(A)
\end{array}$$

commutes up to a functor isomorphism. It follows from Theorem 3.7 that the diagram

$$\begin{array}{ccc}
\Omega BQ\mathbf{P}^{sym}(A)\wedge\Omega BQ\mathbf{P}(A) & \xrightarrow{\mu} & \Omega BQ\mathbf{P}(A) \\
\downarrow\tau & & =\downarrow \\
\Omega BQ\mathbf{P}(A)\wedge\Omega BQ\mathbf{P}^{sym}(A) & \xrightarrow{\mu} & \Omega BQ\mathbf{P}(A)
\end{array}$$

commutes up to a basepoint-preserving homotopy whence the statements c) and the first part of d) of the Theorem. And, finally, the following diagram

$$\begin{array}{ccc} (A\text{-}\mathbf{mod})^{sym} \times \mathbf{P}(A) & \xrightarrow{\otimes_A} & A\text{-}\mathbf{mod} \\ \downarrow \tau & & =\downarrow \\ \mathbf{P}(A) \times (A\text{-}\mathbf{mod})^{sym} & \xrightarrow{\otimes_A} & A\text{-}\mathbf{mod} \end{array}$$

commutes up to a basepoint-preserving homotopy, which proves d). ■

Corollary 3.9 *If the braiding σ is a symmetry, and A is σ-symmetric, then the ring $K_*(A)$ is graded commutative (general case), and $G_*(A)$ is a graded symmetric $K_*(A)$-$K_*(A)$-bimodule (A left Noetherian).*

Proof. $K_*(A) \approx K_*^{sym}(A)$, $G_*(A) \approx G_*^{sym}(A)$. ■

4 Applications

Throughout this section we fix a commutative ring k. All modules and algebras will be automatically assumed to be k-modules and k-algebras. In order to distinguish the K-theory of A-modules in the category $\mathfrak{C}$, and the K-theory of "ordinary" modules, we will refer to a $\mathfrak{C}$-algebra A and its K-theory as $A_{\mathfrak{C}}$ and, respectively, as $K_*(A_{\mathfrak{C}})$.

4.1 Hopf algebras

Let us fix a Hopf algebra H, and let $\mathfrak{C}$ denote the category of left H-modules described in Example 2.1. As usual, we use the following form of the diagonal Δ:

$$\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \in H \otimes H.$$

Definition 4.1 *Given an algebra A in the category of left H-modules, let us denote by $A[H]$ the algebra which equals $A \otimes H$ as a k-module, with the following multiplication:*

$$(b \otimes g) * (a \otimes h) = \sum_i \left(b \cdot g_i^{(1)} a\right) \otimes g_i^{(2)} h.$$

Theorem 4.2 *a) $A[H]$ is an associative algebra with the identity element $1 \otimes 1$.*

*b) The category $A_{\mathfrak{C}}$-**Mod** of left $A_{\mathfrak{C}}$-modules is k-linear equivalent to the category of left $A[H]$-modules.*

Proof. b) Any left $A_{\mathfrak{C}}$-module M has two structures: the left multiplication by the elements of A which will be denoted by $\cdot$, and the left multiplication by the elements of H. Let us define the following $A[H]$-multiplication on M:

$$(a \otimes h) * m := a \cdot hm.$$

Check the associativity condition:

$$(b \otimes g) * [(a \otimes h) * m] = (b \otimes g) * (a \cdot hm) = b \cdot g(a \cdot hm) = \sum_i b \cdot \left(g_i^{(1)} a \cdot g_i^{(2)} hm \right) =$$

$$= \sum_i \left(b \cdot g_i^{(1)} a \right) \cdot g_i^{(2)} hm = \sum_i \left(\left(b \cdot g_i^{(1)} a \right) \otimes g_i^{(2)} h \right) * m = [(b \otimes g) * (a \otimes h)] * m.$$

Conversely, given a $A[H]$-module M, we can define a left A-structure on M by the formula

$$a \cdot m := (a \otimes 1)m,$$

and a left H-structure by

$$hm := (1 \otimes h)m.$$

a) Apply the associativity condition from b) to the $A_{\mathfrak{C}}$-module $A \otimes H$. ■

From now on we assume that H is projective and finitely generated as a k-module.

Theorem 4.3 *Given a braiding element σ in $H \otimes H$, and a σ-commutative algebra $A_{\mathfrak{C}}$, there exists an associative ring structure on the graded Abelian group $K_*(A[H])$, and, if the algebra A is left Noetherian, a $K_*(A[H])$-$K_*(A[H])$-bimodule structure on the graded Abelian group $G_*(A[H])$.*

Proof. Combine Theorems 3.8 and 4.2. ■

Corollary 4.4 *If in the conditions of Theorem 4.3 the braiding σ is a symmetry then the ring $K_*(A[H])$ is graded commutative, and the $K_*(A[H])$-$K_*(A[H])$-bimodule $G_*(A[H])$ is symmetric.*

Proof. Use Corollary 3.9. ■

4.2 G-graded modules

Consider a finite Abelian group G, and a category of G-graded k-modules. Let

$$s : G \times G \longrightarrow U(k)$$

be a bihomomorphism, and let σ be the corresponding braiding (see Example 2.6). Consider a σ-commutative G-graded algebra

$$A_{\mathfrak{C}} = \coprod_{g \in G} A_g.$$

The commutativity means that $ab = s(g,h)ba$ whenever $a \in A_g$ and $b \in A_h$.

Definition 4.5 *Denote by $A(G)$ the algebra of matrices of the form*

$$B = [b_{gh}] : g,h \in G, b_{gh} \in A_{h^{-1}g}$$

with the multiplication:

$$(BC)_{gh} := \sum_{p \in G} b_{gp} \cdot c_{ph}.$$

Theorem 4.6 *There exists an associative ring structure on the graded group $K_*(A(G))$ and, if the algebra A is left Noetherian, a $K_*(A(G))$-$K_*(A(G))$-bimodule structure on the graded Abelian group $G_*(A(G))$.*

Proof. Let us first prove that there exists an isomorphism

$$\varphi : A[H] \approx A(G)$$

where $H = k(G)$. We define φ by the following:

$$(\varphi(a \otimes f))_{gh} := a_{hg^{-1}} f_g$$

where

$$a = \sum_{g \in G} a_g, \text{ and } f = \sum_{p \in G} f_g \theta_g.$$

The statement of the theorem easily follows now from Theorem 4.3. ∎

Corollary 4.7 *Under the conditions of Theorem 4.6, if the function s is anti-symmetric ($s(g,h)s(h,g) = 1$) then the ring $K_*(A(G))$ is graded commutative, and the $K_*(A(G))$-$K_*(A(G))$-bimodule $G_*(A(G))$ is symmetric.*

Proof. Note that σ is a symmetry (Proposition 2.6), and use Corollary 3.9. ∎

4.3 Super-K-theory

Set now $G = \mathbb{Z}_2$, and let

$$s(m,n) = (-1)^{mn} \text{ for } [m],[n] \in \mathbb{Z}_2.$$

The σ-commutative $\mathbb{Z}_2$-graded algebras $A_{\mathfrak{C}}$ are nothing but supercommutative superalgebras, which means that

$$a \cdot b = (-1)^{mn} b \cdot a \tag{3}$$

for $a \in A_{[m]}, b \in A_{[n]}$. The algebra $A(\mathbb{Z}_2)$ can be easily described as the algebra

$$\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

of homogeneous matrices of the form

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$$

where $a_{00}, a_{11} \in A_0$, and $a_{01}, a_{10} \in A_1$.

It is natural to define super-K-theory and super-G-theory as follows:

Definition 4.8

$$K_*^{super}(A) = K_*\left(\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}\right),$$

$$G_*^{super}(A) = G_*\left(\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}\right).$$

We need in the statements c) and d) of the Theorem below that

$$x \cdot x = 0$$

for any $x \in A_1$. It follows from (3) that

$$2\,(x \cdot x) = 0$$

in A_0. That is why we assume in c) and d) that A_0 has no 2-torsion.

Theorem 4.9 *a) If A is supercommutative then there exists a graded commutative ring structure on the graded Abelian group $K_*^{super}(A)$, and, if the algebra A is left Noetherian, a $K_*^{super}(A)$-$K_*^{super}(A)$-bimodule structure on the graded Abelian group $G_*^{super}(A)$. The latter bimodule is symmetric.*

b) There exists a natural homomorphism

$$\varphi_* : K_*(A_0) \oplus K_*(A_0) \longrightarrow K_*^{super}(A).$$

If $A_{\mathfrak{C}}$ is supercommutative, then φ_* is a ring homomorphism.

c) If the algebra $A_{\mathfrak{C}}$ is supercommutative, and A_0 has no 2-torsion, then φ_0 is an isomorphism;

d) If the algebra $A_{\mathfrak{C}}$ is supercommutative, and A_0 has no 2-torsion, then φ_1 is an epimorphism. If, in addition, A is left regular (left Noetherian and of finite left global dimension), then φ_1 is an isomorphism;

e) The homomorphism φ_2 needs not in general be an isomorphism (even in the case of a supercommutative algebra $A_{\mathfrak{C}}$).

Proof. a) Use Corollary 4.7.

b) Consider the algebra A_0 as a superalgebra with a trivial 1-part. The inclusion $A_0 \to A$ of superalgebras induces a homomorphism

$$\varphi_* : K_*(A_0) \oplus K_*(A_0) = K_* \left(\begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix} \right) = K_*^{super}(A_0) \longrightarrow K_*^{super}(A),$$

which in the case of a supercommutative algebra preserves multiplication.

c) Denote by J the nil-radical of A_0, and by B the quotient algebra A_0/J. Define a ring homomorphism

$$\psi : \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix} \longrightarrow \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

by

$$\psi \left(\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \right) = \left(\begin{bmatrix} \overline{a_{00}} & 0 \\ 0 & \overline{a_{11}} \end{bmatrix} \right)$$

The composition

$$\theta_* : K_*(A_0) \oplus K_*(A_0) \longrightarrow K_*^{super}(A) \longrightarrow K_*(B) \oplus K_*(B)$$

is induced by the natural projection

$$A \times A \longrightarrow B \times B.$$

Both kernels

$$\ker \theta = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix} \text{ and } \ker \psi = \begin{bmatrix} J & A_1 \\ A_1 & J \end{bmatrix}$$

are radical ideals, while $\ker \theta$ is nilpotent, and it follows from ([1], Proposition IX (1.3)) that θ_0 is an isomorphism, and ψ_0 is a monomorphism. This implies that all the maps θ_0, φ_0, and ψ_0 are isomorphisms.

d) The proposition above also implies that θ_1 and ψ_1 are epimorphisms. If an element

$$x \in K_1^{super}(A)$$

is represented by a matrix

$$g = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \in GL\left(\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}\right),$$

then one concludes that both a_{00} and a_{11} are invertible matrices over A_0. Indeed, if

$$g^{-1} = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix},$$

then

$$a_{00}b_{00} = 1 - a_{01}b_{10}.$$

The latter matrix is invertible because $a_{01}b_{10}$ is nilpotent. It follows that a_{00} has a right inverse. It can be proved analogously that a_{00} has a left inverse, and therefore is invertible. Similarly for a_{11}. Now we can use elementary row operations and transform g to a block-diagonal matrix

$$\widetilde{g} = \begin{bmatrix} \widetilde{a}_{00} & 0 \\ 0 & \widetilde{a}_{11} \end{bmatrix}$$

One concludes that

$$x \in \operatorname{Im} \varphi_0,$$

and φ_0 is surjective.

Now assume that $A_{\mathfrak{C}}$ is left regular. One can construct an inverse ξ to the homomorphism φ_1: If $x = [g] \in K_1^{super}(A)$ where

$$g = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \in GL\left(\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}\right),$$

let us define

$$\xi(x) := ([a_{00}], [a_{11}]) \in K_1(A_0) \oplus K_1(A_0)$$

We need check that ξ is a homomorphism. Given two elements x and y, one has:

$$\xi(x) = ([a_{00}], [a_{11}]),$$

$$\xi(y) = ([b_{00}], [b_{11}]),$$

$$\xi(xy) = ([a_{00}b_{00} + a_{01}b_{10}], [a_{11}b_{11} + a_{10}b_{01}]).$$

However,

$$a_{00}b_{00} + a_{01}b_{10} = a_{00}b_{00}\left(1 + b_{00}^{-1}a_{00}^{-1}a_{01}b_{10}\right) = a_{00}b_{00}\left(1+\nu\right)$$

where ν is a nilpotent matrix. It means that

$$[a_{00}b_{00}] = [a_{00}b_{00} + a_{01}b_{10}]$$

in the group $K_1^{KV}(A_0) \approx K_1(A_0)$. Similarly for $a_{11}b_{11}$. Therefore, the homomorphism is well defined, a gives a natural splitting for φ. It follows that φ is a monomorphism, and therefore an isomorphism.

e) Set

$$k = A_0 = A_1 = \mathbb{Z},$$

and assume that the multiplication

$$A_1 \otimes A_1 \longrightarrow A_0$$

is zero. It follows from ([12], Theorem 3b) that φ_2 is a split monomorphism, but not an isomorphism. ■

4.4 G-modules

Consider a finite group G, and a Hopf algebra kG. Let $\mathfrak{C} = kG$-**mod**, choose a braiding element

$$\sigma = \sum_{g,h\in G} s_{gh}[g]\otimes[h],$$

and let $A_{\mathfrak{C}}$ be a σ-commutative $\mathfrak{C}$-algebra.

Definition 4.10 *Denote by $A[G]$ the twisted group algebra which consists of formal sums*

$$\sum_{g\in G} a_g[g],\ a_g \in A_g,$$

with a multiplication given by

$$(a_g[g])\,(b_h[h]) := a_g\,(b_h)^g\,[gh].$$

Theorem 4.11 *There exists an associative ring structure on the graded group $K_*(A[G])$ and, if the algebra A is left Noetherian, a $K_*(A[G])$-$K_*(A[G])$-bimodule structure on the graded Abelian group $G_*(A[G])$.*

Proof. It is easy to construct an isomorphism

$$\varphi : A[H] \approx A[G] \; (\varphi(a \otimes [g]) := a[g]).$$

Now apply Theorem 4.3. ■

Corollary 4.12 *Under the conditions of Theorem 4.11, if the braiding σ is a symmetry, then the ring $K_*(A[G])$ is graded commutative, and the $K_*(A[G])$-$K_*(A[G])$-bimodule $G_*(A[G])$ is symmetric.*

Proof. Use Corollary 3.9. ■

References

1. H. Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.
2. P. Deligne. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties.*, volume 900 of *Lecture Notes in Math.*, p. 101–228. Springer, Berlin, 1982.
3. D. R. Grayson. Products in K-theory and intersecting algebraic cycles. *Invent. Math.*, 47(1):71–83, 1978.
4. A. Joyal and R. Street. The geometry of tensor calculus. I. *Adv. Math.*, 88(1):55–112, 1991.
5. V. Lychagin. Braidings and quantizations over bialgebras. Technical Report 61, The Erwin Schrodinger International Institute for Mathematical Physics, November 1993.
6. V. Lychagin. Differential operators and quantizations. Technical Report 44, Institute of Mathematics, University of Oslo, December 1993.
7. V. Lychagin. Braided differential operators and quantizations in ABC-categories. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(9):857–862, 1994.
8. V. Lychagin. Braided differential operators and quantizations in ABC-categories. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(9):857–862, 1994.
9. V. Lychagin. Colour calculus and colour quantizations. *Acta Appl. Math.*, 41(1-3):193–226, 1995.
10. V. Lychagin. Calculus and quantizations over Hopf algebras. *Acta Appl. Math.*, 1998. to appear.
11. S. MacLane. *Categories for the Working Mathematician.* Springer Verlag, 1971.
12. A. V. Prasolov. On Karoubi - Villamayor and Quillen functors for triangular categories. *Trudy Sem. Petrovsk.*, (9):102–110, 1983.
13. D. Quillen. Higher algebraic K-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

14. S. Saavedra. Categories tannakiennes. *Lecture Notes in Math.*, 265, 1972.
15. R. Thomason. Symmetric monoidal categories model all connective spectra. *Theory and Appl. of Categories*, 1(5):78–118, 1995.
16. F. Waldhausen. Algebraic K-theory of generalized free products. I, II. *Ann. of Math. (2)*, 108(1):135–204, 1978.
17. C. A. Weibel. A survey of products in algebraic K-theory. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, p. 494–517. Springer, Berlin, 1981.