Existence of close pseudoholomorphic disks for almost complex manifolds and an application to Kobayashi-Royden pseudonorm

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Abstract

It is proved in the paper[1] that near every pseudoholomorphic disk on an almost complex manifold a disk of almost the same size in any close direction passes. As an application the Kobayashi-Royden pseudonorm for almost complex manifolds is defined and studied.

Introduction

Let (M^{2n}, J) be an almost complex manifold, i.e. $J^2 = -\mathbf{1} \in T^*M \otimes TM$. A mapping $\Phi\,(M_1, J_1) \to (M_2, J_2)$ is called pseudoholomorphic if its differential preserves the complex multiplication in the tangent bundles: $\Phi_* \circ J_1 = J_2 \circ \Phi_*$.

Denote by $e = 1 \in T_0\mathbb{C}$ the unit vector. Let us also denote by D_R the disk in $\mathbb{C}$ of radius R, which is equipped with the standard complex structure J_0. Let $v \in T_pM$, $p = \tau_M v$, with $\tau_M : TM \to M$ being used for the canonical projection. We say that a disk $f : D_R \to M$ passes in the direction v if $f_* e = v$. Due to theorem III from [1] there exists a small pseudoholomorphic disk in the direction of an arbitrary vector v. We study

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non-small pseudoholomorphic disks which lie in a neighborhood of a given pseudoholomorphic disk. The main result is

Theorem 1. *Let a nonconstant pseudoholomorphic disk of radius R pass through a point p at an almost complex manifold (M^{2n}, J):*

$$f_0 : (D_R, J_0) \to (M, J), \qquad (f_0)_*(0)e = v_0 \neq 0.$$

Then for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{V} = \mathcal{V}_\varepsilon(v_0)$ of the vector $v_0 \in TM$ such that in the direction of each vector $v \in \mathcal{V}$ a pseudoholomorphic disk of radius $R - \varepsilon$ passes:

$$f : (D_{R-\varepsilon}, J_0) \to (M, J), \qquad f_*(0)e = v.$$

This theorem has an important application in the theory of invariant metrics. In 1967 Kobayashi [2] introduced a pseudodistance on complex manifolds, which is invariant under biholomorphisms. This gave rise to hyperbolic spaces theory [3–5]. Kobayashi pseudodistance is the maximal pseudodistance among all pseudodistances non-increasing under holomorphic mappings, which on the unit disk $D_1 \subset \mathbb{C}$ coincides with the distance d_D, induced by infinitesimal Poincaré metric in Lobachevskii model

$$dl^2 = \frac{dz\, d\bar{z}}{(1 - |z|^2)^2}.$$

On a complex manifold M the pseudodistance is defined by the formula

$$d_M(p, q) = \inf \sum_{k=1}^{m} d_D(z_k, w_k),$$

where the infimum is taken over all holomorphic mappings $f_k : D_1 \to M$, $k = 1, \ldots, m$, such that $f_1(z_1) = p$, $f_k(w_k) = f_{k+1}(z_{k+1})$ and $f_m(w_m) = q$. In the paper [6] the Kobayashi pseudodistance was extended to the case of arbitrary almost complex manifolds and it was shown that the basic properties of this pseudodistance are preserved.

In 1970 Royden [7] found an infinitesimal analog of the Kobayashi pseudodistance for complex manifolds. We define the corresponding notion in the category of almost complex manifolds and we prove, using theorem 1, the coincidence theorem (theorem 3). We obtain a hyperbolicity criterion (theorem 4). We also consider the reduction procedure, which allows to define geometric invariants of the moduli space for pseudoholomorphic curves.

1. Existence of close pseudoholomorphic disks

1.1. Reformulation of the main result

Let us reformulate theorem 1 using the differential equation language in appropriate coordinates. To begin with choose these coordinates along the disk $f_0(D_R) \subset M$. Due to existence of isothermal coordinates on surfaces [8] the disk $f_0(D_R)$ can be defined in local complex coordinate system $(z^1, \ldots, z^n)$, which is specified in some neighborhood of the disk, via the formulae: $|z^1| \leq R$, $z^2 = \ldots = z^n = 0$. Moreover the disk will be pseudo-holomorphic, $J|_{\mathrm{Im}\, f_0} = J_0$, and $v_0 = (1, 0, \ldots, 0) \in T_0\mathbb{C} \simeq \mathbb{C}$.

Proposition 1. *In an appropriate coordinate system the vector fields $\partial_k = \partial/\partial z^k$ and $\bar{\partial}_k = \partial/\partial \bar{z}^k$ at points of the disk $f_0(D_R)$ satisfy the conditions*

$$J\partial_k = i\partial_k, \quad J\bar{\partial}_k = -i\bar{\partial}_k. \tag{1}$$

Proof. Given equations are already satisfied on the disk for the vector fields ∂_1, $\bar{\partial}_1$. Further at the points of the disk we define transversal to this disk vector fields ∂_k, $\bar{\partial}_k$, $k \geq 2$, in such a way that all the union of $2n$ vectors forms a basis at each point and also that condition (1) is satisfied. Upon constructing the needed vector fields at the points of the disk we extend them to a neighborhood with the help of lemma 1. Obtained structure J coincides with the structure J_0 on the disk and does not necessarily do so outside. $\square$

Lemma 1. *Let we be given k standard commuting vector fields $v_i = \partial_i$, $i = 1, \ldots, k$, and also $n - k$ transversal fields v_j, $j = k+1, \ldots, n$, along the disk $D^k \subset \mathbb{R}^k \times \{0\}^{n-k} \subset \mathbb{R}^n$; at each point $x \in D^k$ all the vectors $v_1, \ldots, v_n$ forming a basis. Then there exist coordinates x^i in a small neighborhood of the disk D^k such that $v_i(x) = \partial_i = \partial/\partial x^i$, $i = 1, \ldots, n$, for all $x \in D^k$.*

Proof. Since the commutators of vector fields along D^k are determined by their 1-prolongations outside D^k, we write the general form for a 1-prolongation of the vector field v_1:

$$v_1 = \sum_{r=1}^{n} \Big(\delta_1^r + \sum_{s=k+1}^{n} x^s \phi_s^r(x_1, \ldots, x_k) \Big) \partial_r \mod \mu^2 \mathcal{D}, \tag{2}$$

where $\mu^2\mathcal{D}$ is the submodule of the module of vector fields, consisting of the vector fields vanishing on the submanifold $D^k \subset \mathbb{R}^n$ to the second order. If on the disk D^k the decomposition of the additional vector fields is written as

$$v_j = \sum_{s=1}^{n} a_j^s(x_1, \ldots, x_k)\partial_s, \; j = k+1, \ldots, n, \tag{3}$$

then the equations $[v_1, v_j] = 0$ with $x^{k+1} = \ldots = x^n$ have the following form

$$\sum_{r=1}^{n}(\partial_1 a_j^r(x^1, \ldots, x^k))\partial_r = \sum_{s=k+1}^{n} a_j^s(x^1, \ldots, x^k)\sum_{r=1}^{n}\phi_s^r(x^1, \ldots, x^k)\partial_r.$$

This system decomposes (by r) on n determinate systems of $n-k$ linear equations with $n-k$ unknowns. The matrix $(a_j^s)_{k+1\le j,s\le n}$ of each system is nondegenerate, hence the system possesses a solution.

Thus the field v_1 is constructed. Let us rectify it: $v_1 = \partial/\partial x^1$. We can assume that on the disk $D^k \subset \{x^{k+1} = \ldots = x^n = 0\}$ the tangent vector fields have the original form $v_i = \partial_i$. In new coordinates the coefficients of the decomposition (3) do not depend on x^1. So one can search for the prolongation of the field v_2 in the form similar to (2), but with no dependence on x^1. Continuing the process we get some coordinates $x^1, \ldots, x^n$, in which the disk D^k belongs to the subspace $\{x^{k+1} = \ldots = x^n = 0\}$ and such that on this disk

$$v_i = \frac{\partial}{\partial x^i}, \; i = 1, \ldots, k, \; v_j = \sum_{s=1}^{n} a_j^s\partial_s, \; a_j^s = \text{const}, \; j = k+1, \ldots, n. \tag{4}$$

Now we prolong the vector fields to a neighborhood via the formula (4). $\square$

Thus we introduce complex coordinates $z^k = x^k + iy^k$ in a neighborhood of the disk $f_0(D^k)$. Now our manifold, being contracted, has the form

$$M_0 = \{|z_1| \le R, \; |z_k| \le R_1, \; k \ge 2\} \simeq D_R \times (D_{R_1})^{n-1} \subset \mathbb{C}^n, \quad R_1 \ll R, \tag{5}$$

and the structure J at points of the disk $D_R = \{(x_1, y_1, 0, \ldots, 0, 0)\}$ has the form

$$J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}, J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}. \tag{6}$$

Writing down Cauchy-Riemann equations $f_* \circ J_0 = J \circ f_*$ on the mapping of the disk $f : (D_{R-\varepsilon}, J_0) \to (M, J)$ (similar to sec. 3.3 from [1]) and using the rectifying conditions (6), we get an equivalent formulation of the main statement ($\partial, \bar\partial$ are considered in the coordinates z^k):

Theorem 1′. *Let n^2 functions $a^i_{\bar m}$ $(i, m = 1, \dots, n)$ of the class $C^{k+\lambda}$, $k \in \mathbb{Z}_+$, $\lambda \in (0,1)$ be given on a manifold M_0 of the form (5). Let $\varepsilon \in (0, R)$ be an arbitrary small real number. If $a^i_{\bar m}(z) = 0$ for all points $z \in D_R \times \{0\}^{n-1} \subset M_0$, then the equation*

$$\bar\partial z^i + \sum_{m=1}^{n} a^i_{\bar m}(z) \bar\partial \bar z^m = 0, \quad z^i(0) = p^i, \quad \partial z^i(0) = u^i, \quad i = 1, \dots, n,$$

has a solution $z^i = z^i(\zeta) \in C^{k+1+\lambda}(D_{R-\varepsilon}; M_0)$ subject to the restriction that the neighborhood $\mathcal{V} = \mathcal{V}(v_0) \ni v$ of the vector $v_0 = (1, 0, \dots, 0) \in T_0 M_0$ is chosen sufficiently small. Here $v = (p, u)$, $p = \tau_M v \in M_0$, $u \in T_p M_0$.

1.2. Covering of the neighborhood by disks and another reformulation

Theorem 1″. *One can set $p = 0$ in the formulation of theorem 1′. Thus in the chosen coordinates the equation on the pseudoholomorphic disk sought for takes the following form:*

$$\begin{cases} \bar\partial z^1 & = & -\sum_{m=1}^{n} a^1_{\bar m}(z) \bar\partial \bar z^m, \\ \bar\partial z^I & = & -\sum_{m=1}^{n} a^I_{\bar m}(z) \bar\partial \bar z^m. \end{cases} \tag{7}$$

$$z^1(0) = 0, z^I(0) = 0, \ (\partial z^1(0), \partial z^I(0)) = (u^1, \dots, u^n),$$

where the multiindex I stands for $(2, \dots, n)$.

Next statement shows equivalence of theorems 1′ and 1″.

Proposition 2. *For every pseudoholomorphic disk $f_0 : D_R \to M_0$ and every $\varepsilon > 0$ a small neighborhood of the image $f_0(D_{R-\varepsilon})$ can be covered by the images of close pseudoholomorphic disks f of radii $R - \delta$, where $\delta < \varepsilon$: $\mathcal{O}(\operatorname{Im} f_0(D_{R-\varepsilon})) \subseteq \bigcup_f \operatorname{Im} f(D_{R-\delta})$.*

Proof. Perturb the almost complex structure J in a neighborhood of the disk $f_0(D_R)$ so that it coincides with the standard integrable structure J_0 near the boundary of this neighborhood: for every $\varepsilon > 0$ there exists such an almost complex structure $\tilde J$ that $\tilde J = J$ in a small neighborhood of the disk $f_0(D_{R-\varepsilon/2}) = D_{R-\varepsilon/2} \times \{0\}^{n-1} \subset M_0$ and $\tilde J = J_0$ in a neighborhood of the boundary of the manifold $\tilde{M} = D_R \times (D_\delta)^{n-1} \subset M_0 \subset \mathbf{C}^n$. Further one can suppose that $\tilde{M} \subset \hat{M} = S^2_R \times (S^2_{2R})^{n-1} \simeq (S^2)^n$, where $\hat{M}$ can be equipped with an almost complex structure $\hat{J}$, which coincides with $\tilde{J}$ in $\tilde{M}$ and which equals the standard integrable structure J_0 in the complement. Let us supply the manifold $\hat{M}$ with the symplectic structure $\omega = \omega_0^{(1)} \oplus \omega_0^{(2)} \oplus \ldots \oplus \omega_0^{(2)}$, where ω_0 is the standard volume form, and also $\omega_0^{(1)}(S^2_R) = \pi R^2$ and $\omega_0^{(2)}(S^2_{2R}) = 4\pi R^2$. Decreasing if necessary the size of the neighborhood of the disk $f_0(D_{R-\varepsilon/2})$ we can suppose that the almost complex structure $\hat{J}$ is tamed by the symplectic structure ω, i.e. $\omega(\xi, \hat{J}\xi) > 0$ for $\xi \neq 0$.

Denote by $A \in H_2(\hat{M}; \mathbb{Z})$ the homology class of the sphere $S^2_R \times \{*\}^{n-1} \subset \hat{M}$. The disk $f_0(D_{R-\varepsilon/2})$ can be extended to the entire rational pseudoholomorphic curve $u_0 : S^2 \to \hat{M}$, which lies in the class A. Let us consider the space $\mathcal{M}(A, \hat{J}\,)$ of entire pseudoholomorphic curves $u : S^2 \to \hat{M}$ of the class A. Since the class A cannot be decomposed into a sum of homology classes $\sum_{i=1}^n A_i$, $n \geq 2$, with $\omega(A_i) > 0$, then Gromov compactness theorem ([9] sec. 1.5.B or [10] Sec. 4.3.2) implies the compactness of the space $\mathcal{M}(A, \hat{J}\,)/G$, where $G \simeq PSL_2$ is the complex automorphisms group of the sphere (S^2, J_0), $\dim G = 6$. Moreover for almost complex structure $\hat{J}$ of general position the space $\mathcal{M}(A, \hat{J}\,)$ is a smooth manifold of dimension $2n+4$ [9] sec. 2.1–2.2; [10] sec. 3.1.2. Consider the space of nonparametrized pseudoholomorphic curves $\mathcal{W}(A, \hat{J}\,) = \mathcal{M}(A, \hat{J}\,) \times_G S^2$. This space is a compact manifold of dimension $2n$. Let us consider the evaluation map $e : \mathcal{W}(A, \hat{J}\,) \to \hat{M}$, which is defined by the formula $e(u, z) = u(z)$ for $z \in S^2$, $u \in \mathcal{M}(A, \hat{J}\,)$. We suppose the group G acts on $\mathcal{M} \times S^2$ by conjugation, $\phi(u, z) = (u \circ \phi^{-1}, \phi(z))$, $\phi \in G$, whence the correctness of the definition for e. Since A-curves foliate the manifold $\hat{M}$ outside a small neighborhood of the image $u_0(S^2)$ (because there $\hat{J} = J_0$), the map e has degree 1, $\deg e = 1$. Therefore through every point, close to the curve $u_0(S^2)$, some pseudoholomorphic curve $u(S^2)$ passes, which is homologous to the curve $u_0(S^2)$.

To eliminate the general position condition for $\hat{J}$ we take a sequence $\hat{J}_k$ of the general position almost complex structures, which tends to $\hat{J}$ in C^∞-topology, and use the compactness theorem [10]B.4.2. Intersecting the obtained set of pseudoholomorphic spheres $u(S^2)$ with a small neighborhood $\mathcal{O}$ of the disk $f(D_{R-\varepsilon/2})$, we get the desired set of the disks $f(D)$ in a neighborhood $(\mathcal{O}, J)$. Smoothness of these disks follows from the standard elliptic regularity [1] sec. 5.4, 4.3; [10] sec. B.4.1. $\square$

1.3. Spaces, norms and estimates

Define the λ-Hölder norm of complex-valued functions on the disk D_R of radius R by the formula $\|f\| = |f| + (2R)^\lambda H_\lambda[f]$, $\lambda \in (0,1)$, where

$$H_\lambda[f] = \sup_{w\neq 0}\left|\frac{f(z+w)-f(z)}{w^\lambda}\right|, \qquad |f| = \sup|f(z)|.$$

The space $C^\lambda(D_R, M_0)$ of λ-Hölder maps consists of all maps $f : D_R \to M_0$, the components of which have finite λ-norms, $\|f^i\| < \infty$. The space $C^{k+\lambda}(D_R, M_0)$, $k \in \mathbb{Z}_+$, of $(k+\lambda)$-Hölder maps consists of all maps, the partial derivatives of which up to the k-th order inclusive belong to C^λ.

Let us also introduce the space $B = C^{k+\lambda}_\bullet(D_R, M_0)$ consisting of all maps $f \in C^{k+\lambda}$, $f(0) = 0$, with the norm $\|f\| = \sum_1^n \|f^i\|$. Note that the space B can be also supplied with the norm

$$\|f\|' = \max\{\|\partial f\|, \|\bar\partial f\|\}.$$

Proposition 3. *The spaces* $(C^\lambda, \|\cdot\|)$ *and* $(C^{1+\lambda}_\bullet, \|\cdot\|')$ *are Banach.*

The second statement follows from the first and the estimate [1] 7.1.c–7.1.e

$$\|f\| \le 6R\|f\|'. \tag{8}$$

Consider the Cauchy operators

$$Sf(w) = \frac{1}{2\pi i}\oint_{\partial D_R}\frac{f(\zeta)}{\zeta - w}\,d\zeta, \qquad Tf(w) = \frac{1}{2\pi i}\iint_{D_R}\frac{f(\zeta)}{\zeta - w}\,d\zeta\wedge d\bar\zeta.$$

Recall the basic properties of these operators [1] 6.1–6.2:

$$f \in C^\lambda(D) \Rightarrow Tf \in C^{1+\lambda}(D),\ \bar\partial Tf = f, \tag{9}$$

$$f \in C^\lambda(D) \Rightarrow Sf \in C^\lambda(\mathrm{Int}\, D),\ \bar\partial Sf = 0,\ STf = 0, \tag{10}$$

$$f = Sf + T\bar\partial f \quad \text{(Cauchy-Green-Pompéiu formula)}\ , \tag{11}$$

$$\|Tf\|' \le c_1\|f\|, \quad \|Sf\| \le c_2\|f\|, \tag{12}$$

Consider also the operators $T_k f(w) = Tf(w) - \sum_{s=0}^k \frac{1}{s!}\partial^s Tf(0)w^s$.

Lemma 2. *For the points $w \in \operatorname{Int} D$ the following formula holds*

$$T_k f(w) = \frac{w^{k+1}}{2\pi i} \iint_{D_R} \frac{f(\zeta)}{(\zeta - w)\zeta^{k+1}} \, d\zeta \wedge d\bar{\zeta}.$$

Lemma 3. *The operator $T_\infty = \lim_{k\to\infty} T_k$ is defined for functions $f \in C^\lambda(D_R)$, and moreover $T_\infty f \in C^{1+\lambda}(D_{R-\varepsilon})$.*

Lemma 4. $T_k(w^l \bar{w}^m) = \left[\begin{array}{ll} \dfrac{w^l \bar{w}^{m+1}}{m+1}, & l < k+m+2, \\ \dfrac{w^l \bar{w}^{m+1}}{m+1} - \dfrac{R^{2(m+1)}}{m+1} w^{l-m-1}, & l \geq k+m+2. \end{array} \right.$

Corollary. $T_\infty(w^l \bar{w}^m) = w^l \bar{w}^{m+1}/(m+1)$.

Thus the operator T_∞ represent the integration by $\bar{\zeta}$ of the polynomials on D_R. Let us also besides the space B consider its closed subset $B_\delta = \{f = (f_1, \ldots, f_n) \in B, \, |f_1 - \zeta| \leq \delta, \, |f_k| \leq \delta, \, k \geq 2\}$. We will seek a solution f of the Cauchy-Riemann equation (7) in the space B_δ for a small neighborhood $\mathcal{V}$ of the vector v_0.

1.4. Proof of theorem 1″

Idea of the proof. Equation (7) was solved in the paper [1], theorem III, where the velocity vector v was fixed and the radius $R \ll 1$ of the disk was supposed small. For this the linearization of almost complex structure at the point was considered. Because of the proximity of equations on pseudoholomorphic curves for the given almost complex structure J and for the linearized one J_0 the following map was contractible:

$$\Phi : B \to B, \quad (\Phi f)^i(\zeta) = v^i \zeta + T_1 \left(-\sum_m a^i_{\bar{m}}(f) \bar{\partial} \bar{f}^m \right)(\zeta). \tag{13}$$

In our situation radius of the disk is not small, therefore the word for word carrying over the arguments from [1] is possible only if the structure J differs from J_0 on the disk f_0 by a second order smallness quantity, i.e. if the functions $a^i_{\bar{1}}$ on the disk f_0 as well as their derivatives vanish. In general situation it is not the case, so we linearize the almost complex structure J along the disk f_0. Here the linearization is parametrized by the coordinate

$z^1 = \zeta$ along this disk. Solutions of complex linear equation behave similarly to solutions of the real equation $\dot{x} = Ax + B$: for non-small values of the parameter it is false that $e^{At} \approx 1$, so the terms of the series $e^{At} = \sum_{s=0}^{\infty}(At)^s/s!$ are not absolute decreasing, but this property becomes true beginning with some number $s \geq s_0$. Thus finite sums of the series for the exponent does not form a contracting sequence, yet to achieve this one should consider the sums beginning with some big number.

Let us turn to the proof. Similarly to [1], starting with formulae (11), (10), we seek a solution of equation (7) in the form (13), but we replace the space B by B_δ. In fact, as noted above, we should change the definition of the operator Φ to improve the convergence. Let us consider the automorphism of the space $\mathbf{C}^n$, which comes from the contraction of the space M_0,

$$z^1 \to z^1, \; z^I \to \frac{z^I}{N}, \; N \gg 1. \tag{14}$$

Since $a^i_{\bar{m}} = 0$ along the disk $D_R \times \{0\}^{n-1} \subset M_0$, the function $a^1_{\bar{1}}$ becomes small and the functions $a^I_{\bar{1}}$ become very close to their linearizations by the variables z^I in the norm $\|\cdot\|'$ for large N in equation (7).

Consequently the first equation of (7), considered as one z^I-parametric equation, can be solved by the iteration method, when we use formula (13) for complex dimension 1 and change B to B_δ. For small δ and big N in (14) the estimates from [1] sec. 5.2 yield the contractibility of the iteration procedure in the norm $\|\cdot\|'$. This iteration procedure will be denoted by $z^1 \mapsto \Psi^1(z^1, z^I)$.

To consider the second equation of (7) let us linearize the functions used in it by z^I:

$$a^I_{\bar{1}}(z) = \sum_{m \geq 2} \left(a^I_{\bar{1};m}(z^1) z^m + a^I_{\bar{1};\bar{m}}(z^1) \bar{z}^m \right) + \hat{a}^I_{\bar{1}}(z). \tag{15}$$

In this formula the functions $\hat{a}^I_{\bar{1}}(z)$ have the second order of smallness along the disk $D_R \subset M_0$. Let us also set $\hat{a}^I_{\bar{m}}(z) = a^I_{\bar{m}}(z)$ when $m \neq 1$.

According to Weierstrass theorem the coefficients at linear by z^I terms in (15) are approximated by polynomials depending on z^1, $\bar{z}^1$ in the norm $|\cdot|$ on D_R:

$$a^I_{\bar{1};m}(z^1) = p^I_m(z^1, \bar{z}^1) + \alpha^I_m(z^1), \; a^I_{\bar{1};\bar{m}}(z^1) = p^I_{\bar{m}}(z^1, \bar{z}^1) + \alpha^I_{\bar{m}}(z^1), \; |\alpha^I_m|, |\alpha^I_{\bar{m}}| < \varepsilon.$$

Let $A^I(\zeta, z^I) = \sum_{m\geq 2}(p^I_m(\zeta,\bar\zeta)z^m + p^I_{\bar m}(\zeta,\bar\zeta)\bar z^m)$, $A^I_\delta(\zeta, z) = A^I(\zeta, z^I) - A^I(z^1, z^I)$. Then the second equation of (7) can be written in the form

$$\bar\partial z^I(\zeta) = -A^I(\zeta, z^I) + U^I(z(\zeta)), \tag{16}$$

where the summands of the remainder $U^I = A^I_\delta + U^I_1 + U^I_2 + U^I_3$ have the form

$$U^I_1(z) = A^I(z^1, z^I)(1 - \bar\partial \bar z^1), \quad U^I_2(z) = -\sum_m \hat a^I_{\bar m}\bar\partial \bar z^m,$$

$$U^I_3(z) = -\sum_{m\geq 2}(\alpha^I_m(z^1)z^m + \alpha^I_{\bar m}(z^1)\bar z^m)\bar\partial \bar z^1.$$

We approximate equation (16) by the following equation with linear by z^I right hand size and polynomial by ζ coefficients:

$$\bar\partial z^I(\zeta) = -A^I(\zeta, z^I). \tag{17}$$

A solution of this equation can be constructed as the limit of the iteration procedure

$$z^I_{(k+1)} = v^I\zeta - T_\infty[A^I(\zeta, z^I_{(k)})]. \tag{18}$$

By the corollary of lemma 4 the iteration of integration by means of the operator T_∞ has the form $T^k_\infty(w^l\bar w^m) = w^l m!\,\bar w^{m+k}/(m+k)!$, which implies that the iteration process (18) converges under any initial condition $z^I_{(0)}$ to a solution of equation (17), and moreover the convergence is exponential. In particular, beginning with some number k, the sequence $z^I_{(k)}$ is contractible. And what is more there exist constants C and μ, depending only on almost complex structure J (i.e. on coefficients $a^i_{\bar m}$), such that for every $k\geq 1$ and polynomial $p(\zeta,\bar\zeta)$ the following inequality holds:

$$\|T^k_\infty[A^I(\zeta, p)]\|' \leq Ce^{\mu R}\|p\|. \tag{19}$$

We now define the iteration procedure to compute $z^I(\zeta)$. Let the iterative term $z^I_{[r]}$ be already constructed. Additionally in virtue of the previous step the given term is equal to the sum of a polynomial $P^I_{[r]}(\zeta,\bar\zeta)$ and a function $\theta^I_{[r]}(\zeta)\in C^{1+\lambda}$. Represent the last function by Weierstrass theorem as the sum of a polynomial (by $\zeta,\bar\zeta$) and an error: $\theta^I_{[r]}(\zeta) = Q^I_{[r]}(\zeta) + q^I_{[r]}(\zeta)$, $|q^I_{[r]}|\leq \nu|\theta^I_{[r]}|$. Define the next term by the formula

$$z^I_{[r+1]}(\zeta) = v^I\zeta - T_\infty[A^I(P^I_{[r]})] - T^{k_r}_\infty[A^I(Q^I_{[r]})] - T_1[A^I(q^I_{[r]})] + T_1[U^I(z_{[r]})].$$

Here $A^I = A^I(\zeta, \cdot)$ and k_r is such a number that beginning with number k_r the sequence $T^k_\infty[A^I(Q^I_{[r]})]$ contracts with the coefficient ε_r. In addition (cf. (8)) the following estimates for the additional terms take place:

$$\|A^I_\delta(\zeta, z') - A^I_\delta(\zeta, z'')\| \le c_3\delta\|z' - z''\|', \ \|U^I_1(z') - U^I_1(z'')\| \le c_3\delta\|z' - z''\|',$$
$$\|U^I_2(z') - U^I_2(z'')\| \le c_3\delta\|z' - z''\|', \ \|U^I_3(z') - U^I_3(z'')\| \le c_4\varepsilon\|z' - z''\|'.$$

Taking inequality (12) into account we conclude that for small δ, ε, ε_r and ν the sequence $z^I_{[r]}$ is contractible: $\|z^I_{[r+1]} - z^I_{[r]}\|' \le (1-\kappa)\|z^I_{[r]} - z^I_{[r-1]}\|'$ for some $\kappa < 1$ independent of r. Therefore, taking into consideration the iteration by Ψ^1 for the variable z^1, we get a convergent in $C^{1+\lambda}$ sequence, the limit of which has to be the desired solution. Actually, set $z^1_{[r+1]} = \Psi^1(z^1_{[r]}, z^I_{[r]})$, taking as parameter z^I the iterative term $z^I_{[r]}$. In what follows in determination of the term $z^I_{[r+1]}$ we assume $z^1 = z^1_{[r]}$. Thus we obtain the sequence $z_{[r]}$.

Due to the estimates considered and inequality (19) the terms and the limit of the sequence $z_{[r]}$ differ from its initial term $z_{[0]} = v\zeta$ less than exponentially by R with respect to $|v - v_0|$ in the norm $\| \cdot \|'$. Therefore for small $|v - v_0| \ll 1$ all the terms and the limit of the iterative sequence lie in B_δ. Hence the sequence converges in B_δ. Now it is easily seen that the limit of the sequence $z_{[r]}$ is a solution of the equation (7). When the coefficients have smoothness $a^i_{\bar{m}} \in C^{k+\lambda}(M_0)$, then the obtained solution, which is of smoothness $C^{1+\lambda}$, will be actually of higher smoothness class $C^{k+1+\lambda}$. This follows from the standard elliptic regularity methods for our equation [1] 5.4, 4.3, [10] B.4.1. When $a^i_{\bar{m}} \in C^\infty(M_0)$ we get a smooth solution of the Cauchy-Riemann equation $z(\zeta) \in C^\infty(D_{R-\varepsilon}; M_0)$. $\square$

1.5. Jet spaces and connection with h-principle

Let us call the foliation by pseudoholomorphic disks any embedding (immersion) $\Phi : D_R \times N^{2n-2} \to M$ such that all the mappings $\Phi|_{D_R\times\{x\}}$ are pseudoholomorphic and the image of the map Φ covers the entire manifold M. The construction of proposition 2 together with the positivity of intersections in dimension 4 ([9] 2.1.C$_2$; [11] 1.1) imply

Proposition 4. *Let (M, J) be a four-dimensional almost complex manifold. For every embedded (immersed) pseudoholomorphic disk $f : D_R \to M$ and every $\varepsilon > 0$ small neighborhood of the image $f(D_{R-\varepsilon})$ allows the foliation by pseudoholomorphic disks.*

Let us consider the manifold of pseudoholomorphic jets $\mathcal{J}^1_{PH}(D_R; M)$ of the mappings $u : D_R \to M$. Its points are triples (ζ, z, Φ), where $\zeta \in D_R$, $z \in M$, and $\Phi : (T_\zeta D_R, J_0) \to (T_z M, J(z))$ is a complex linear mapping. It was shown in the paper [13] that the manifold $\mathcal{J}^1_{PH}$ possesses a canonical almost complex structure $J_{[1]}$, which is equal to $J_0 \oplus J \oplus J$ regarding the induced by some minimal connection decomposition $T_p\mathcal{J}^1_{PH} = T_\zeta D_R \oplus T_z M \oplus T_p\mathcal{F}$, where $\mathcal{F}$ is the fiber of the natural projection $\tau : \mathcal{J}^1_{PH}(D_R, M) \to D_R \times M$. The canonical projection $\pi : \mathcal{J}^1_{PH}(D_R; M) \to M$ is pseudoholomorphic and any pseudoholomorphic mapping $f : D_R \to M$ lifts canonically to the pseudoholomorphic mapping $j^1 f : D_R \to \mathcal{J}^1_{PH}(D_R; M)$, $j^1 f(\zeta) = (\zeta, f(\zeta), d_\zeta f)$.

We define the structure $J_{[1]}$ in a different way (cf. [14], remark 1). If $p = (\zeta, z, \Phi) \in \mathcal{J}^1_{PH}$, we can assume that the mapping Φ is the differential at the point ζ of some small pseudoholomorphic disk $u : D_\varepsilon \to M$. Denote by $p^{(2)}$ the 2-jet of the disk u at the point $\zeta \in D_\varepsilon \subset D_R$. Consider the map $j^1 u : D_\varepsilon \to \mathcal{J}^1_{PH}$. The tangent space at the point p depends only on the value $p^{(2)}$. Denote this tangent space by $L_{p^{(2)}}$.

Consider the natural projection $\rho : \mathcal{J}^1_{PH} \to D_R$ with the fiber $\mathcal{H}$. We have $T_p\mathcal{J}^1_{PH} = L_{p^{(2)}} \oplus T_p\mathcal{H}$, both summand being naturally equipped with complex structures. Set $J_{[1]} = J_0 \oplus J$. This structure does not depend on the choice of $p^{(2)}$, i.e. it is defined canonically.

Definition. Let us call a pseudoholomorphic disk $g : D_R \to \mathcal{J}^1_{PH}(D_R, M)$ *holonomic*, if the mapping g is the 1-jet lifting of some pseudoholomorphic disk from D_R to M: $g = j^1 f$.

Proposition 2 applied to a holonomic disk $g = j^1 f : D_R \to \mathcal{J}^1_{PH}$ yields existence of a pseudoholomorphic disk g' through each point arbitrary close to the image of the disk g, which however needs not be a holonomic disk, $g' \neq j^1(\pi \circ g')$. In this sense theorem 1 provides a more strong statement. Actually, closeness of initial points of the disks $g = j^1 f$ and $g' = j^1 f'$ in $\mathcal{J}^1_{PH}(D_{R-\varepsilon}; M)$ means closeness of initial points and initial directions of the maps f and f' in TM. Thus theorem 1 implies existence of C^1-close disk f', and we can set $g' = j^1 f'$. Thus we proved

Theorem 2. *Through every point, which is close to the image of embedded (immersed) holonomic pseudoholomorphic disk $g : D_R \to \mathcal{J}^1_{PH}(D_R; M)$, an embedded (immersed) holonomic pseudoholomorphic disk $g' : D_{R-\varepsilon} \to \mathcal{J}^1_{PH}(D_{R-\varepsilon}; M)$ passes.* $\square$

In other words, proposition 2 remains also valid in the holonomic situation. The statement just proved is a particular case of the so-called h-principle [15]. It is also interesting to get the holonomic version of proposition 4.

2. Kobayashi-Royden pseudonorm

2.1. Definition of the pseudonorm and its main properties

Let us consider the set $\mathcal{R}(v) = \bigcup_{r>0} \mathcal{R}_r(v)$, where $\mathcal{R}_r(v)$ for $r \in \mathbb{R}_+$ consists of pseudoholomorphic mappings $f : D_1 \to M$, such that $f_*(0)e = rv$.

Definition. Let us call the Kobayashi-Royden pseudonorm on an almost complex manifold M the function on the tangent bundle TM, which is defined by the formula

$$F_M(v) = \inf_{\mathcal{R}(v)} \frac{1}{r}.$$

According to theorem III from [1] the set $\mathcal{R}_r(v)$ is nonempty for small r, so the definition is correct. We call the function F_M pseudonorm since it is nonnegative and homogeneous of degree one: $F_M(tv) = |t| F_M(v)$. However F_M can vanish in some directions and the triangle inequality does not hold. The next statement follows from the very definition.

Proposition 5. *Given any vector $v \in TM_1$ and any pseudoholomorphic mapping $f : (M_1, J_1) \to (M_2, J_2)$ we have*

$$F_{M_2}(f_* v) \leq F_{M_1}(v).$$

Let us fix some norm $|\cdot|$ on TM.

Proposition 6. (i) *There exists a constant C_K for every compact $K \subset M$ such that each vector $v \in TM$ with $\tau_M v \in K$ satisfies*

$$F_M(v) \leq C_K |v|.$$

(ii) *Let M be a compact manifold (with possible boundary) equipped with an almost complex structure J, which is tamed by an exact symplectic form*

$\omega = d\alpha$, $\omega(\xi, J\xi) > 0$ *for* $\xi \neq 0$. *Then there exists such a constant* $c_M > 0$, *that for all* $v \in TM$

$$F_M(v) \geq c_M |v|.$$

Proof. For a small neighborhood U of the point $p \in M$ the estimates of sec. 5.2a of the paper [1] imply existence of a number $\varepsilon > 0$, dependent only on the almost complex structure J and the neighborhood U, such that for every $q \in U$, $v \in T_qM$, $|v| = 1$, and $r \in (0, \varepsilon)$ there exists a pseudoholomorphic disk $f : D_1 \to M$ such that $f(0) = q$, $f_*(0)e = rv$. Setting $C_U = 1/\varepsilon$ we have $F_M(v) \leq C_U|v|$ for all (now not necessarily unit) vectors v for which $\tau_M v \in U$. Since a compact set can be covered by a finite number of neighborhoods U, the first statement of the proposition is proved. The second part is a reformulation of the nonlinear Schwarz lemma [9] 1.3.A: if an almost complex structure J on a compact manifold is tamed by an exact symplectic structure ω, then the derivative at zero of any pseudoholomorphic disk $f : D_1 \to M$, passing through a fixed point at the manifold, is bounded by a non-depending on the disk constant: $|f_*(0)e| < C$. $\square$

Proposition 7. *The function* F_M *is upper semicontinuous.*

Proof. The inequality $\overline{\lim\limits_{v \to v_0}} F_M(v) \leq F_M(v_0)$ is equivalent to the statement of theorem 1 because $F_M(v) = \inf(1/R)$, where the lower bound is considered over all mappings $f : D_R \to M$, such that $f_*(0)e = v$. $\square$

2.2. Coincidence theorem

Define a function $\bar{d}_M : M \times M \to \mathbb{R}$ by the formula

$$\bar{d}_M(p, q) = \inf_{\gamma} \int_0^1 F_M(\dot{\gamma}(t))\, dt,$$

where the lower bound is taken over all piecewise smooth paths γ from the point p to q. Propositions 6(i) and 7 imply correctness of the definition and

Proposition 8. *The function* $\bar{d}_M$ *is pseudodistance.* $\square$

Theorem 3. *Introduced pseudodistance coincides with the Kobayashi pseudodistance,* $d_M = \bar{d}_M$.

Proof. The inequality $\bar{d}_M \le d_M$ is evident because $F_M(v) = \inf|\xi|$, where the lower bound is taken over all pseudoholomorphic mappings $f : D_1 \to M$, $f_*\xi = v$, and the norm is count with respect to the Poincaré metric. Let us prove the reverse. We follow the Royden's proof [7].

Let γ be a smooth curve from a point p to a point q such that $\int_\gamma F_M < \bar{d}_M(p,q)+\varepsilon$. Due to upper semicontinuity there exists a continuous on $[0,1]$ function h, such that $h(t) > F_M(\dot\gamma(t))$ and

$$\int_0^1 h(t)\,dt < \bar{d}_M(p,q)+\varepsilon,$$

i.e. for sufficiently dense partition $0 = t_0 < t_1 < \ldots < t_k = 1$ we have

$$\sum_{i=1}^k h(t_{i-1})(t_i - t_{i-1}) < \bar{d}_M(p,q)+\varepsilon.$$

Consider arbitrary pseudoholomorphic curve $u_t^\gamma : D_\delta \to M$, which satisfies the conditions $u_t^\gamma(0) = \gamma(t)$ and $(u_t^\gamma)_* e = \dot\gamma(t)$. Define for small $\Delta t \in \mathbb{R}_+ \subset \mathbb{C}$ the curve $\hat\gamma(t;\Delta t) = u_t^\gamma(\Delta t)$. Since $\hat\gamma(t;\Delta t) = \gamma(t+\Delta t) + O(|\Delta t|^2)$, propositions 8 and 6 imply that for small Δt it holds:

$$\begin{aligned} d_M(\gamma(t),\gamma(t+\Delta t)) &\le d_M(\gamma(t),\hat\gamma(t;\Delta t)) + d_M(\hat\gamma(t;\Delta t),\gamma(t+\Delta t)) \\ &\le F_M(\dot\gamma(t))\Delta t + O(|\Delta t|^2) \le (1+\varepsilon)h(t)\Delta t. \end{aligned}$$

Thus for sufficiently dense partition

$$d_M(p,q) \le \sum_{i=1}^k d_M(\gamma(t_{i-1}),\gamma(t_i)) < (1+\varepsilon)(\bar{d}_M(p,q)+\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary constant, the theorem is proved. $\square$

2.3. Hyperbolicity and nonhyperbolicity

Definition. Almost complex manifold (M,J) is called *hyperbolic* if the pseudodistance d_M is a distance.

Let us consider the unit tangent vectors bundle $\tau_M^{(1)} : T_1M \to M$ for some norm $|\cdot|$, and let $F_M^{(1)} : T_1M \to \mathbb{R}$ be the restriction of the Kobayashi-Royden pseudonorm to it. Proposition 6(i) and theorem 3 imply

Theorem 4. *The function $F_M^{(1)}$ is bounded on compact subsets in M. Manifold M is hyperbolic iff $F_M^{(1)}$ is bounded away from zero on compact subsets.*

Now let us consider the case of nonhyperbolic manifold M, for example let it possess pseudoholomorphic spheres. In the case of general position for the almost complex structure J, which is tamed by some symplectic form ω on M, the set of all pseudoholomorphic spheres in a fixed homology class $A \in H_2(M;\mathbb{Z})$ (completed for compactness by the set of decomposable rational curves) is a finite-dimensional manifold $\mathcal{M}(A;J)$ [9, 10]. We define by the *reduction procedure* some pseudodistance on this manifold. Namely for any two pseudoholomorphic spheres $f_i : S^2 \to M$, defined up to holomorphical reparametrization of S^2 let

$$d_{\mathcal{M}}([f_1],[f_2]) = d_M(p_1,p_2),$$

where $p_i \in \mathrm{Im}(f_i)$ are arbitrary points on the images. It is easily seen that $d_{\mathcal{M}}$ is correctly defined pseudodistance on the manifold $\mathcal{M}$.

As an example note that the defined pseudodistance $d_{\mathcal{M}}$ is a distance for almost complex manifold $M^4 = \Sigma_g^2 \times S^2$ with $g > 1$, where the structure J is tamed by the standard product symplectic form: as in proposition 4 one proves that M^4 is fibered by pseudoholomorphic spheres and there is an isomorphism $\mathcal{M} \simeq \Sigma_g^2$. However in the case of four-dimensional manifolds this definition is of importance only in the case of zero self-intersection. Actually if $A \cdot A > 0$ (for nonexceptional case $A \cdot A \geq 0$ [11]), then two spheres $\mathrm{Im}(f_1)$ and $\mathrm{Im}(f_2)$ of the given homology class do intersect. Thus $d_{\mathcal{M}}([f_1],[f_2]) = 0$.

It was shown in the paper [16] that for N large enough the manifold $\mathcal{M} \times \mathbb{R}^{2N}$ possesses a homotopically canonical almost complex structure $\tilde{J}$. Kobayashi pseudodistance $d_{\mathcal{M}\times\mathbb{R}^{2N}}$ induces a pseudodistance $\hat{d}_{\mathcal{M}}$ on $\mathcal{M}$ via reduction over $\mathbb{R}^{2N}$. In this connection there arises a natural question of existence of almost complex structures $\tilde{J}$ such that the pseudodistances $\hat{d}_{\mathcal{M}}$ and $d_{\mathcal{M}}$ coincide.

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