Symplectic and contact Lie algebras with an application to Monge-Ampère equations

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Abstract

In this paper* we consider symplectic and contact Lie algebras. We define contactization and symplectization procedures and describe its main properties. We also give classification of such algebras in dimensions 3 and 4. The classification in dimension 4 is closely connected with normal forms of nondegenerate elliptic equations of the second order on two-dimensional surfaces with transitive symmetry group in first jets. We point out this connection and discuss normal forms.

Introduction

A Lie-Poisson group G is a Lie group equipped with a Poisson structure such that if we extend the Poisson structure naturally to the product $G \times G$ then the Lie multiplication map $G \times G \to G$ is Poisson. Since the structures are invariant we may equivalently talk of Lie-Poisson algebras. When the Poisson structure is nondegenerate we call it symplectic. Thus we obtain the problem of description of symplectic Lie algebras.

In this paper we define and study symplectic and contact Lie algebras. For general manifolds in [1] the procedure of symplectization and contactization was defined which is a functor between the categories of exact symplectic and contact manifolds. We study the corresponding notions for contact and exact symplectic Lie algebras. Note that the invariance condition forces us

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to change the general constructions. We also give a description of contact in dimension 3 and symplectic in dimension 4 Lie algebras.

Recall ([4]) that a Lie bialgebra is a Lie algebra $\mathcal{G}$ with a Lie algebra structure on $\mathcal{G}^*$, these structures being compatible. There is a bijective correspondence between Lie-Poisson algebras and Lie bialgebras. When Lie-Poisson algebra is nondegenerate, i.e. symplectic, we obtain that Lie algebra structures on $\mathcal{G}$ and $\mathcal{G}^*$ are equivalent or that the map $\mathcal{G} \to \mathcal{G} \wedge \mathcal{G} \subset \mathcal{G} \otimes \mathcal{G}$ dual to the Lie multiplication on $\mathcal{G}^*$ is a 1-cocycle. Thus every symplectic Lie algebra gives rise to a solution of the classical Yang-Baxter equation ([4]).

Another application of symplectic Lie algebras occurs in dimension 4 where they correspond to invariant under some Lie group transitive action elliptic equations depending on two variables. To be more exact we talk about Monge-Ampère equations on the plane and about the generalization of them given in [7]; we called the corresponding equations generalized Monge-Ampère. For such equations the equivalence problem was solved in [6]. Namely for every nondegenerate elliptic equation there was constructed some canonical $\{e\}$-structure and the defining Monge-Ampère equation structures proved to have a canonical expression by means of this $\{e\}$-structure. Thus the recovering procedure for Monge-Ampère equation by its invariant is close to algorithmic. Basing on the classification of 4-dimensional symplectic Lie algebras we may give normal forms of nondegenerate elliptic generalized Monge-Ampère equations.

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1. Symplectic and contact Lie algebras

It is natural to consider left-invariant structures on Lie groups. For in this case they may be treated as structures on Lie algebras. For example the de Rham's complex for left-invariant differential forms on a Lie group G coincides with the complex for Lie algebra $\mathcal{G}$ cohomologies' complex with coefficients in the trivial module $\mathbb{R}$. Hence everything may be expressed in the language of algebras (and re-presented for groups).

1.1. The definitions and examples of nonexistence

Let us consider a Lie algebra $\mathcal{G}$ and the corresponding cohomology complex with differentials $d : C^i \to C^{i+1}$ (see [5], [10]), $C^i = \wedge^i \mathcal{G}^*$.

Definition 1. Lie algebra $\mathcal{G}$ of even dimension $2n$ is called symplectic when equipped with a nondegenerate closed 2-form $\omega \in C^2$: $d\omega = 0$, $\omega^n \neq 0 \in C^{2n}$. Lie algebra $\mathcal{G}$ of odd dimension $2n+1$ is called contact when equipped with a 1-form $\alpha \in C^1 = \mathcal{G}^*$ such that $\alpha \wedge (d\alpha)^n \neq 0 \in C^{2n+1}$.

Let us also give an equivalent definition (the only difference for contact structures is that Π determines the projective class $\{s\alpha \mid s \in \mathbb{R}^*\}$).

Definition $1'$**.** A closed 2-form ω is called symplectic if the mapping $\mathcal{G} \to \mathcal{G}^*$, $v \mapsto i_v\omega$, is an isomorphism (of linear spaces). A Lie algebra is called contact if it possesses a codimension 1 linear subspace Π such that for none vector $w \in \Pi \setminus \{0\}$ it holds $ad_w\Pi \subset \Pi$.

Now the description of Lie groups with left-invariant symplectic (contact) structure is reduced to the description of symplectic (contact) Lie algebras. Not every even-dimensional Lie algebra is symplectic (the same for contact of course, take the commutative one). For example let $\mathcal{G}$ be a compact Lie algebra (whence reductive, see [2]). Then it cannot possess an exact symplectic form, for this is obvious for (compact) Lie groups in view of Stokes' formula. Thus for compact algebras with $H^2(\mathcal{G}) = 0$ we derive they are not symplectic. Thanks to the Künneth's formula this is the case with compact semisimple or compact reductive with 1-dimensional center Lie algebras. Here we used the Whitehead lemmas ([5]): for semisimple Lie algebra $\mathcal{H}$ we have $H^1(\mathcal{H}; \mathbb{R}) = H^2(\mathcal{H}; \mathbb{R}) = 0$.

Now the compactness restriction above may be easily got over. Let us remind that a Lie algebra $\mathcal{G}^n$ is called unitary ([11]) if $H_n(\mathcal{G}; \mathbb{R}) \neq 0$. Equivalently for some (=every) basis e_i of $\mathcal{G}$ and the corresponding basis e_i^* of $\mathcal{G}^*$ we have $\sum de_i^*(e_i, \cdot) = 0$. As examples we have reductive and nilpotent Lie algebras. For unitary algebras the Poincaré duality holds: $H^k(\mathcal{G}) \simeq H_{n-k}(\mathcal{G})$.

Theorem 1. *Unitary Lie algebra $\mathcal{G}$ possesses no exact symplectic form.*

Proof. If the symplectic form ω on $\mathcal{G}$ with $\dim \mathcal{G} = 2n$ is exact, $\omega = d\alpha$, then the differential of the form $\alpha \wedge \omega^{n-1}$ is nontrivial and hence $H^{2n}(\mathcal{G}) = 0$, which contradicts the Poincaré duality since $H_0(\mathcal{G}) = \mathbb{R}$. $\square$

Corollary. *If for a unitary Lie algebra $\mathcal{G}$ the second cohomology group is trivial, $H^2(\mathcal{G}) = 0$, then it is not symplectic. These are the cases with semisimple and reductive with 1-dimensional center Lie algebras.* $\square$

1.2. Unification of symplectization and contactization

The following construction coincides with symplectization for even dimension and contactization for odd.

Definition 2. The genre of a differential form $\alpha \in \Omega^1(M)$ is the highest degree of a nonzero form in the sequence

$$\alpha, d\alpha, \alpha \wedge d\alpha, (d\alpha)^2, \alpha \wedge (d\alpha)^2, \ldots$$

Let us call a pair (M^n, α) nondegenerate if the genre of α equals $n = \dim M$.

For even n we get an exact symplectic manifold with the structure $d\alpha$ and for odd we get a contact one with the structure form α (to be exact a contact structure is a distribution $\operatorname{Ker} \alpha$, but up to a double covering any contact structure may be obtained from a contact form and keeping in mind Lie algebra constructions we see that these two ways are equivalent).

Now we define *the suspension*, i.e. a method to construct an $(n+1)$-dimensional nondegenerate manifold (M_+, α_+) by a nondegenerate (M, α). It is called symplectization in odd case and contactization in the even (see [1]). Let $M_+^{n+1} = M^n \times \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} \backslash \{0\} = \mathbb{R}_- \cup \mathbb{R}_+$ (one may take instead $\mathbb{R}_+$). Let t be a parameter on $\mathbb{R}^*$ and define the form α_+ to be $t\alpha + dt$, where we consider α to be naturally lifted to M_+.

Proposition 1. *The pair (M_+, α_+) is nondegenerate.*

<u>*Proof.*</u> $d\alpha_+ = dt \wedge \alpha + t d\alpha$. Hence for $n = 2k$ we have $\alpha_+ \wedge (d\alpha_+)^n = t^n dt \wedge (d\alpha)^n \neq 0$. And for $n = 2k+1$ $(d\alpha_+)^{n+1} = t^n dt \wedge \alpha \wedge (d\alpha)^n \neq 0$. $\square$

Another way to obtain this construction is to consider M_+ as a 1-dimensional bundle over M with the connection form α_+ and the curvature form $d\alpha$.

Now we may define the prolongations of the structure diffeomorphisms (contactizations of symplectomorphisms and symplectizations of contactomorphisms) and so on.

1.3. Symplectizations and contactizations of Lie algebras

The differential constructions considered above are not invariant and hence must be a little changed for Lie algebras. It turns out that every time there exists some contactization while symplectization sometimes cannot take place. By the direct sum beyond we mean the sum as vector spaces not algebras. Let us denote by Z^i the subspace $\operatorname{Ker} d$ of C^i.

Let us recall that Lie algebra structure can be fixed in two equivalent ways: by the commuting relations $[e_i, e_j] = c_{ij}^k e_k$ in Lie algebra or by the Maurer-Cartan equations $df^k = -\frac{1}{2} c_{ij}^k f^i \wedge f^j$ in its dual, where f^i stands for the dual basis to e_i. In addition the Jacobi identities are equivalent to the integrability conditions $d^2 = 0$.

1.3.1. Suspensions

Let $\mathcal{G}^{n+1} = \mathcal{H}^n \oplus \mathbb{R}^1$, $\mathcal{H}$ being an ideal (it is also possible to consider the case of subalgebras). Let us suppose that $(\mathcal{H}, \alpha)$ is a nondegenerate pair. Set $\alpha_+ = \pi^*\alpha$ for the projection $\pi : \mathcal{G} \to \mathcal{H}$ induced from the direct decomposition.

Definition 3. The pair $(\mathcal{G}, \alpha_+)$ is called *a suspension* over $(\mathcal{H}, \alpha)$ if it is a nondegenerate pair. For the even-dimensional case it is also called a contactization, for the odd one — a symplectization.

Since $\mathcal{H}$ is an ideal the Lie algebra structure on $\mathcal{G}$ is given by an outer derivation $[A] \in H^1(\mathcal{H}, \mathcal{H})$. Here $[A]$ is a class of the derivation $A \in \operatorname{Der}(\mathcal{H})$ modulo the subalgebra of inner derivations $\{ad_w \mid w \in \mathcal{H}\}$ and the derivation A stands for ad_v with v a generator of $\mathbb{R}^1$.

Note that under the change of the transversal line $\mathbb{R}^1$, $v \mapsto v + w$, $w \in \mathcal{H}$, the suspension form is changed by the formula $\alpha_+ \mapsto \alpha_+ - \alpha(w)v^*$, where $v^* \in (\mathbb{R}^1)^*$ is the dual to v covector, $v^*(\mathcal{H}) = 0$, $v^*(v) = 1$. Thus it seems that the set of parameters for the suspension is the derivations $A \in \operatorname{Der}(\mathcal{H})$ themselves, not their classes.

Let us take now a geometrical point of view.

Contactization. Let $(\mathcal{H}^{2n}, d\alpha)$ be a symplectic Lie algebra, $\alpha \in \mathcal{H}^*$. Let $\Pi^{2n-1} = \operatorname{Ker}\alpha$, u be some transversal vector to it in $\mathcal{H}$ and $w \in \operatorname{Ker}(d\alpha|_\Pi) \setminus$

$\{0\}$. The condition $\alpha \wedge (d\alpha)^n \neq 0$ on $\mathcal{G}$ is equivalent to $d\alpha(v,w) \neq 0$ or $ad_v w \notin \Pi$. Note that the addition to $A = ad_v$ an inner derivation ad_x, $x \in \Pi$, does not change the form α_+. Moreover the derivation $A = ad_u$ enjoys the property $Aw \notin \Pi$. Thus we have proved

Proposition 2. *The set of all contactizations is nonempty and is parametrized by the set*

$$\{A \in \mathrm{Der}(\mathcal{H}) \,|\, Aw \notin \Pi\}/\{ad_x \,|\, x \in \Pi\}. \qquad \square$$

Note that the contactization may also be applied for any other potential $\alpha + \beta$, $d\beta = 0$, of the symplectic form $\omega = d\alpha$.
Symplectization. Let $(\mathcal{H}^{2n-1}, \alpha)$ be a contact Lie algebra, $\alpha \in \mathcal{H}^*$. Let $\Pi^{2n-2} = \mathrm{Ker}\,\alpha$ and $w \in \mathrm{Ker}\,d\alpha \setminus \{0\}$. The condition $(d\alpha)^n \neq 0$ on $\mathcal{G}$ is equivalent to $d\alpha(v,w) \neq 0$ or $ad_v w \notin \Pi$. Note that the addition to $A = ad_v$ an inner derivation ad_x, $x \in \Pi$, does not change the form α_+. On the other hand the addition of ad_w does change the form: $\alpha_+ \mapsto \alpha_+ - \alpha(w)v^*$, $\alpha(w) \neq 0$. However since $\mathcal{H}$ is an ideal $dv^* = 0$ and the form $\omega = d\alpha_+$ is preserved. As in the symplectization we take it for the main object we have proved

Proposition 3. *The set of all symplectizations is parametrized by the set*

$$\{[A] \in H^1(\mathcal{H},\mathcal{H}) \,|\, Aw \notin \Pi\}. \qquad \square$$

Note that the symplectization sometimes cannot take place. Actually let $H^1(\mathcal{H},\mathcal{H}) = 0$ (as it is in the semisimple case). Then every derivation is inner and for $[A] = 0$ we may take representative $A = 0$ for which $Aw = 0 \in \Pi$.

1.3.2. Ideals of codimension 1 in symplectic algebras

Let us consider another way to obtain a symplectic structure on a Lie algebra $\mathcal{G}^{2n} = \mathcal{H}^{2n-1} \oplus \mathbb{R}^1$, $\mathcal{H}$ being an ideal. The Lie algebra structure is given by the class $[A] \in H^1(\mathcal{H},\mathcal{H})$ of the derivation $A = ad_v$, $v \in \mathbb{R}^1$. Let's assume $\mathcal{H}$ is equipped with a closed 2-form with $\mathrm{rk}\,\omega = 2n-2$. Extend it to $\mathcal{G}$ by the formula $i_v\omega = \alpha$, $\alpha \in \mathcal{H}^*$. Since $d\omega|_{\mathcal{H}} = 0$ the closeness takes the form

$$0 = i_v d\omega = L_v\omega - d i_v\omega = A\omega - d\alpha$$

(both the action and the differential live on $\mathcal{H}$). Let $w \in \mathrm{Ker}(\omega|_{\mathcal{H}}) \setminus \{0\}$. The nondegeneracy of ω on $\mathcal{G}$ $\alpha \wedge \omega^{n-1} \neq 0$ is equivalent to $w \notin \Pi^{2n-2} = \mathrm{Ker}\,\alpha$. Note that changing of A by an inner derivation ad_x, $x \in \mathcal{H}$, preserves the condition $A\omega = d\alpha$ because under this transformation $\alpha \mapsto \alpha + i_x\omega$ and $di_x\omega = L_x\omega$ on $\mathcal{H}$. The condition $\alpha(w) \neq 0$ is also preserved since $\omega(w, x) = 0$ for all $x \in \mathcal{H}$. However we can modify α by closed forms $\alpha \mapsto \alpha + \beta$, $d\beta = 0$. Let's call the described construction also the symplectization. We have proved

Proposition 4. *The set of all symplectizations of an odd-dimensional Lie algebra equipped with maximally nondegenerate closed 2-form $\omega \in \wedge^2\mathcal{H}^*$ is parametrized by the set*

$$\{[A] \in H^1(\mathcal{H}, \mathcal{H}),\ \alpha \in \mathcal{H}^* \,|\, A\omega = d\alpha,\ \alpha \wedge \omega^{n-1} \neq 0\}. \qquad \square$$

Note that if ω is a differential of a contact form $\omega = d\gamma$, $\gamma \wedge (d\gamma)^{n-1} \neq 0$, then the construction just described coincides with the symplectization from 1.3.1.

Corollary. *In symplectic Lie algebra there does not exist a semisimple ideal of codimension 1.*

Proof. In this case $[A] \in H^1(\mathcal{H}, \mathcal{H}) = 0$, one can take $A = 0$. Thus $d\alpha = 0$ which according to $H^1(\mathcal{H}) = 0$ implies $\alpha = 0$ and $\alpha \wedge \omega^{n-1} = 0$. $\square$

1.4. Classification in dimensions less than 5

The first nontrivial dimension is 3 and in it almost all algebras are contact. In what follows let $\mathcal{G}'$ denote the commutator of $\mathcal{G}$. In the next theorem we use definition $1'$.

Theorem 2. *Every three-dimensional Lie algebra is contact save for the commutative algebra and the Lie algebra $\langle x, y, z \,|\, [x, z] = x, [y, z] = y, [x, y] = 0\rangle$. The contact structure is unique with the exception of Lie algebra $sl(2)$ where we have two contact structures (up to an isomorphism).*

Proof. According to definition $1'$ a contact structure is a 2-dimensional subspace which is not a subalgebra. For $\dim \mathcal{G}' = 3$ we have two simple Lie algebras $so(3)$ and $sl(2)$. For $so(3)$ the Lie multiplication is given by the vector product and all 2-planes are equivalent. For $sl(2)$ if Π^2 is not a subalgebra let $\langle z \rangle = (\Pi^2)'$. Note that the choice of Π^2 gives us the orientation of the space $\mathcal{G}^3$. Actually for any bivector $x \wedge y \in \wedge^2 \Pi^2$ it is given by the 3-vector $x \wedge y \wedge [x, y]$. Now ad_z is an automorphism of Π^2. As it is a derivation it is divergence-free, $\mathrm{Tr}(ad_z) = 0$, and we may assume $\det(ad_z) = 1$. Thus $\mathrm{Sp}(ad_z) = \{\pm i\}$ and the claim follows. If $\dim \mathcal{G}' = 2$, $\mathcal{G}'$ is commutative and hence $\dim(L) = 1$ for $L = \Pi^2 \cap \mathcal{G}'$. If for a transversal to $\mathcal{G}'$ vector z $ad_z|_{\Pi} = \mathrm{const} \cdot \mathbf{1}$ the Lie algebra $\mathcal{G}$ is not contact. Otherwise L^1 may be any line in $\mathcal{G}'$ different from eigenspaces of ad_z and the classification follows. In the last case $\dim \mathcal{G}' = 1$ the plane Π^2 is arbitrary transversal to $\mathcal{G}'$ and to the center so that Π^2 is unique up to isomorphism. $\square$

Remark. We just showed that Lie algebra $sl(2)$ possesses two contact structures. The Killing form $k(X, Y)$ is a nondegenerate quadratic form of the signature $(2, 1)$. So there exists a cones of isotropic vectors in $sl(2)$ and we distinguish the contact structures subject to possibilities of k being nondegenerate on $\Pi^2 = \mathrm{Ker}\, \alpha$ or having (two) isotropic directions. The case of one isotropic direction corresponds to Π^2 being a subalgebra. Any 2-subspace is equally well characterized by the orthogonal 1-subspace $(\Pi^2)^{\perp}$ with respect to the Killing form. We may assume that the orientation on $sl(2)$ is given by the 3-vector $X_0 \wedge X_1 \wedge X_2$ with

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have two possibilities: the contact structure α is *positive* $\alpha \wedge d\alpha > 0$ which corresponds to $k(X, X) > 0$ for every $X \in (\Pi^2)^{\perp} \setminus \{0\}$ and the contact structure α is *negative* $\alpha \wedge d\alpha < 0$ which corresponds to $k(X, X) < 0$.

The spherization ST^*M^2 of the cotangent bundle of every surface of genus $g > 0$ can be obtained as quotient of the group $\mathrm{Sl}_2(\mathbb{R})$ by a discrete subgroup. Actually this follows from isomorphism $\mathrm{Sl}_2(\mathbb{R})/\{\pm 1\} \simeq ST^*L^2$, L^2 being the Lobachevskii plane (see [3] for details). This isomorphism agrees with the orientation fixed above if we define the orientation on T^*M^2 by the canonical symplectic form: $dv = \frac{1}{2}\omega^2 = dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2$. Now the quotient procedure above gives us two contact structures on ST^*M^2: positive α_+ which is the standard pdq and negative α_- which is the connection form associated with a metric of constant negative curvature.

Next we consider symplectic algebras of dimension 4.

Lemma 1. *There does not exist a four-dimensional Lie algebra $\mathcal{G} = \mathcal{G}'$.*

Proof. Let $\mathcal{G} = \mathcal{H} \oplus R$ be a Levi decomposition, $\mathcal{H}$ being a semisimple subalgebra and R being a radical. We have $\dim R = 1$. There is an action of $\mathcal{H}$ on R. Since $\mathcal{H}$ is simple the action is trivial (otherwise the kernel is an ideal). Thus we have direct Lie summation $\mathcal{G} = \mathcal{H} \oplus R$ and $\mathcal{G}' = \mathcal{H}' = \mathcal{H}$. □

The classification of 4-dimensional structures may be obtained from the Bianchi's classification of 3-dimensional algebras and the suspension method from 1.3.2. The proof of the following theorem uses a more direct method. It seems rather technical but this is due to the fact that it almost coincides with the classification of four-dimensional Lie algebras (which may be extracted from the proof).

Theorem 3. *A four-dimensional Lie algebra $\mathcal{G}$ is symplectic iff it has the form (we list them according to the dimension of $\mathcal{G}'$):*

1. $\dim \mathcal{G}' = 3$. *Here an algebra is symplectic iff $H^4(\mathcal{G}) = 0$. More precisely, the structure Maurer-Cartan equations have the form $de_4^* = 0$, $de_3^* = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$ and*
   - (i) $de_1^* = \lambda e_1^* \wedge e_4^*$, $de_2^* = (1-\lambda) e_2^* \wedge e_4^*$, $\lambda \neq 0, 1$.
   - (ii) $de_1^* = \frac{1}{2} e_1^* \wedge e_4^* + \lambda e_2^* \wedge e_4^*$, $de_2^* = -\lambda e_1^* \wedge e_4^* + \frac{1}{2} e_2^* \wedge e_4^*$.
   - (iii) $de_1^* = \frac{1}{2} e_1^* \wedge e_4^* + \kappa e_2^* \wedge e_4^*$, $de_2^* = \frac{1}{2} e_2^* \wedge e_4^*$.
2. $\dim \mathcal{G}' = 2$. *In this case the structure equations are $de_3^* = de_4^* = 0$ and*
   - (i) $de_1^* = e_1^* \wedge e_3^* + e_2^* \wedge e_4^*$, $de_2^* = e_2^* \wedge e_3^* + \nu_1 e_1^* \wedge e_4^* + \nu_2 e_2^* \wedge e_4^*$.
   - (ii) $de_1^* = e_1^* \wedge e_3^*$, $de_2^* = e_3^* \wedge e_4^*$.
   - (iii) $de_1^* = e_2^* \wedge e_3^*$, $de_2^* = e_3^* \wedge e_4^*$.
   - (iv) $de_1^* = e_1^* \wedge e_3^*$, $de_2^* = -e_2^* \wedge e_3^*$.

(v) $de_1^* = e_2^* \wedge e_3^*$, $de_2^* = -e_1^* \wedge e_3^*$.

3. $\dim \mathcal{G}' = 1$. *Here* $de_2^* = de_3^* = de_4^* = 0$ *and*

(i) $de_1^* = e_1^* \wedge e_2^*$.

(ii) $de_1^* = e_2^* \wedge e_3^*$.

4. Lie algebra $\mathcal{G}$ *is commutative,* $\mathcal{G}' = 0$.

In addition in cases 1 and 2(i) *the symplectic structure can be chosen exact and the set of such structures is nonempty Zarissky-open in* $\mathrm{Im}\, d|_{C^1} \subset C^2$, *in particular everywhere dense. In other cases the symplectic form is unique (in 2*(ii) *up to the sign) and is not exact. In cases 2*(ii)-(iii) *the structure has the form* $\omega = e_1^* \wedge e_3^* + e_2^* \wedge e_4^*$, *and in cases 2*(iv)-(v), *3 and 4 it is* $\omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$.

Proof. **1°**. $\dim \mathcal{G}' = 3$. Thanks to lemma 1 there exists a covector e_4^* such that $de_4^* = 0$. First consider the case all exact 2-forms df^* are degenerate, i.e. decomposable. Then there exists a 3-dimensional subspace $\Pi^3 \subset \mathcal{G}^*$ such that $dC^1 \subset \wedge^2 \Pi^3 \subset C^2$.

Consider the case $e_4^* \in \Pi^3$. One may complete e_4^* to a basis e_i^* such that $\Pi^3 = \langle e_2^*, e_3^*, e_4^* \rangle$ and $de_2^* = f^* \wedge e_4^*$, with $f^* \in \langle e_2^*, e_3^* \rangle$. Up to a change of basis there are two possibilities: $f^* = e_2^*$, $de_3^* \notin \langle e_2^*, e_3^* \rangle \wedge e_4^*$ or $de_3^* = h^* \wedge e_4^*$, $h^* \in \langle e_2^*, e_3^* \rangle$. In the first case one chooses a covector e_1^* satisfying $de_1^* = e_3^* \wedge e_4^*$. Then $de_3^* = ae_2^* \wedge e_3^* + v^* \wedge e_4^*$, $a \neq 0$, and the equation $d^2 e_1^* = 0$ does not hold. In the second case the covector e_1^* can be chosen to satisfy the equality $de_1^* = e_2^* \wedge e_3^*$. Let us consider a linear automorphism of the plane $\langle e_2^*, e_3^* \rangle$, given by the formulas: $e_2^* \mapsto f^*$, $e_3^* \mapsto h^*$. The equation $d^2 e_1^* = 0$ implies its trace vanishes. Moreover eigenvalues are nonzero and are complex conjugated if not real. Thus after a transformation $de_2^* = e_2^* \wedge e_4^*$ and $de_3^* = -e_3^* \wedge e_4^*$, or $de_2^* = e_3^* \wedge e_4^*$ and $de_3^* = -e_2^* \wedge e_4^*$. It's easy to see that in every case the Lie algebra is not symplectic and moreover $H^4(\mathcal{G}) \neq 0$.

Let now $e_4^* \notin \Pi^3$. We may assume that Π^3 is generated by covectors e_1^*, e_2^* and e_3^*, i.e. there is a direct Lie decomposition $\mathcal{G} = \mathcal{H} \oplus \mathbb{R}^1$, $\mathcal{H}$ being a 3-dimensional Lie algebra. We may choose the basis in Π^3 in such a manner that the symplectic form $\omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$. Then $d\omega = 0$ implies $de_3^* = 0$. Thus $\dim \mathcal{G}' < 3$. According to Bianchi's classification there are exactly two

3-dimensional algebras with the commutator $\mathcal{H} = \mathcal{H}' = \mathcal{G}'$: $\mathcal{H} = sl(2)$ and $\mathcal{H} = so(3)$ and in every case $H^4(\mathcal{G}) = H^3(\mathcal{H}) = \mathbb{R}$.

Let us complete the classification of symplectic algebras $\mathcal{G}$ of the type $\mathcal{H} \oplus \mathbb{R}$. We have: $de_1^* = A_1 e_1^* \wedge e_2^* + B_1 e_1^* \wedge e_3^* + C_1 e_2^* \wedge e_3^*$, $de_2^* = B_2 e_1^* \wedge e_3^* + C_2 e_2^* \wedge e_3^*$, $de_3^* = de_4^* = 0$. The condition $d\omega = 0$ gives $B_1 + C_2 = 0$ and the conditions $d^2 e_i^* = 0$ are equivalent to the relations $A_1 B_1 = A_1 B_2 = 0$. Assume first $A_1 = 0$. Considering the canonical forms of linear automorphism of the plane $\langle e_1^*, e_2^* \rangle$, given by the formula $e_i^* \mapsto de_i^*(\cdot, e_3^*)$, we get the Lie algebras 2(iv), 2(v) and 3(ii). The other case $A_1 \neq 0$ leads to the Lie algebra 3(i). Let us call further the considered case 3-dimensional.

Now let us consider the possibility of nondegenerate form df^*, $f^* \in \mathcal{G}^*$. It is clear that if Lie algebra $\mathcal{G}$ possesses an exact symplectic form df^*, then for a Zarissky-open set of covectors $f^* \in \mathcal{G}^*$ the 2-form df^* is symplectic.

We may assume that there exists a basis e_i^* such that $de_3^* = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$. Actually this is equivalent to the existence of a covector $X \notin \langle e_4^* \rangle$ and a number $\lambda \neq 0$ such that $\sigma = dX - \lambda X \wedge e_4^*$ is a decomposable 2-form, i.e. $0 = \sigma^2 = (dX)^2 - \lambda dX \wedge X \wedge e_4^*$. For almost every X the form $(dX)^2$ is nondegenerate. So the pair (X, λ) does exist iff there exists a covector X such that $dX \wedge X \wedge e_4^* \neq 0$. Let us suppose this 4-form is zero for every X. Let us consider some generic covector $X = e_1^* \notin \langle e_4^* \rangle$. Then by our arguments there are covectors e_2^*, e_3^* with the property $de_1^* = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$. The differentials of the other basis covectors must have the form $de_2^* = e_2^* \wedge X_1 + e_4^* \wedge X_2$, $de_3^* = e_3^* \wedge X_3 + e_4^* \wedge X_4$, $X_i \in \langle e_1^*, e_2^*, e_3^* \rangle$. From the equation $d^2 e_1^* = 0$ we have: $X_1 = \mu e_1^*$, $X_2 = \nu e_1^* + \kappa e_3^*$, $X_3 = \kappa e_1^* + e_2^*$. Thus $de_2^* = -\mu e_1^* \wedge e_2^* - \nu e_1^* \wedge e_4^* - \kappa e_3^* \wedge e_4^*$ and from $d^2 e_2^* = 0$ we get $\nu = 0$, i.e. as $d(e_1^* \wedge e_2^*) \neq 0$ $de_2^* = -\mu de_1^*$ which contradicts the assumption $\dim \mathcal{G}' = 3$.

Now let us consider the general form for the differentials (we may assume $A_2 = 0$):

$$\begin{aligned} de_1^* &= A_1 e_1^* \wedge e_2^* + B_1 e_1^* \wedge e_3^* + C_1 e_1^* \wedge e_4^* + D_1 e_2^* \wedge e_3^* + E_1 e_2^* \wedge e_4^* + F_1 e_3^* \wedge e_4^*, \\ de_2^* &= \phantom{A_1 e_1^* \wedge e_2^* +{}} B_2 e_1^* \wedge e_3^* + C_2 e_1^* \wedge e_4^* + D_2 e_2^* \wedge e_3^* + E_2 e_2^* \wedge e_4^* + F_2 e_3^* \wedge e_4^*. \end{aligned}$$

The condition $d^2 e_3^* = 0$ implies $F_1 = F_2 = 0$, $C_1 + E_2 = 1$, $B_1 + D_2 = 0$. From $d^2 e_2^* = 0$ we have $B_2 A_1 = C_2 A_1 = 0$, and

$$-B_2 C_1 - B_2 + C_2 B_1 - D_2 C_2 + E_2 B_2 = 0, \ -B_2 E_1 + C_2 D_1 - D_2 = 0. \quad (\dagger)$$

If $A_1 \neq 0$ then $B_2 = C_2 = D_2 = 0$ and $d^2 e_1^* = 0$ gives $E_2 = 0$ which contradicts $de_2^* \neq 0$. Thus $A_1 = 0$. Consider the automorphism of the plane

$\langle e_1^*, e_2^* \rangle$ given by the formula $e_i^* \mapsto de_i^*(\cdot, e_3)$ and consider its canonical forms (according to the equations above it is traceless).

a). $B_1 = -D_2 = \lambda$, $D_1 = B_2 = 0$. Then (†) implies $\lambda = 0$.

b). $B_1 = B_2 = D_2 = 0$, $D_1 = \lambda$. If $\lambda \neq 0$ then (†) implies $C_2 = 0$. Now the equation $d^2 e_1^* = 0$ implies $E_2 = 0$ which is impossible since $de_2^* \neq 0$.

c). $B_1 = D_2 = 0$, $B_2 = -D_1 = \lambda$. Let us suppose $\lambda \neq 0$. Then (†) implies $E_1 = -C_2$, $C_1 = 0$, $E_2 = 1$. Now the equation $d^2 e_1^* = 0$ implies $\lambda = 0$.

Thus we have $B_i = D_i = 0$. Hence we can consider the automorphism of the plane $\langle e_1^*, e_2^* \rangle$ given by the formula $e_i^* \mapsto de_i^*(\cdot, e_3)$ and its canonical forms. Since the trace of this automorphism is $C_1 + E_2 = 1$ we obtain all cases 1(i)-(iii).

2°. $\underline{\dim \mathcal{G}' = 2}$. For some basis we have $de_3^* = de_4^* = 0$ and differentials de_1^* and de_2^* are linear independent and have a general form:

$$
\begin{aligned}
de_1^* &= A_1 e_1^* \wedge e_2^* + B_1 e_1^* \wedge e_3^* + C_1 e_1^* \wedge e_4^* + D_1 e_2^* \wedge e_3^* + E_1 e_2^* \wedge e_4^* + F_1 e_3^* \wedge e_4^*,\\
de_2^* &= B_2 e_1^* \wedge e_3^* + C_2 e_1^* \wedge e_4^* + D_2 e_2^* \wedge e_3^* + E_2 e_2^* \wedge e_4^* + F_2 e_3^* \wedge e_4^*.
\end{aligned}
$$

From the conditions $d^2 e_i^* = 0$ we have:

$$
\begin{gathered}
A_1F_1 - B_1E_1 + C_1D_1 - E_2D_1 + E_1D_2 = 0,\\
-A_1F_2 - D_1C_2 + E_1B_2 = 0, \ -B_2C_1 + B_1C_2 - C_2D_2 + B_2E_2 = 0, \qquad (\diamondsuit)\\
A_1B_2 = A_1C_2 = A_1D_2 = A_1E_2 = 0.
\end{gathered}
$$

If $A_1 \neq 0$ then $B_2 = C_2 = D_2 = E_2 = F_2 = 0$ and $de_2^* = 0$, i.e. $\dim \mathcal{G}' = 1$. Thus $A_1 = 0$.

Let us consider an automorphism of the plane $\langle e_1^*, e_2^* \rangle$ given by the formula $e_i^* \mapsto de_i^*(\cdot, e_3) = B_i e_1^* + D_i e_2^* (\mathrm{mod}\, e_4^*)$, $i = 1, 2$. Further we consider all possible canonical forms of this automorphism. The repeated phrase "we may assume" means "there exists a coordinate transformation such that".

1. The eigenvalues $\lambda_1 \neq \lambda_2$. Let, say, $\lambda_1 \neq 0$. We may assume $B_1 = \lambda_1$, $B_2 = 0$, $D_1 = 0$, $D_2 = \lambda_2$. From the conditions $(\diamondsuit)$ we have: $C_2 = E_1 = 0$. We may assume $C_1 = F_1 = 0$.

1.1. $E_2 \neq 0$. By a transformation is reduced to case 2(i), $\nu_1 = 0$, $\nu_2 = 1$.

1.2. $E_2 = 0$, $\lambda_2 \neq 0$. By a transformation is reduced to the 3-dimensional case considered above.

1.3. $E_2 = \lambda_2 = 0$. In this case $F_2 \neq 0$ and we get case 2(ii).

2. Jordan box with nonzero eigenvalue: $B_1 = 1$, $B_2 = 0$, $D_1 = \mu$, $D_2 = 1$. From the conditions ($\diamondsuit$) we have: $C_2\mu = 0$, $(C_1 - E_2)\mu = 0$.

2.1. $\mu = 0$. We may assume $F_1 = F_2 = 0$. A transformation in the plane $\langle e_3^*, e_4^*\rangle$ leads to $C_1 = 0$.

2.1.1. $E_1 \neq 0$. In this case we have equations 2(i).

2.1.2. $E_1 = 0$, $C_2 \neq 0$. A transformation leads to case 2(i).

2.1.3. $E_1 = C_2 = 0$, $E_2 \neq 0$. By a transformations is reduced to 2(i), $\nu_1 = 0$, $\nu_2 = 1$.

2.1.4. $C_2 = E_1 = E_2 = 0$. We get the 3-dimensional case considered above.

2.2. $\mu \neq 0$. Hence $C_2 = 0$, $C_1 = E_2$. A transformation in the space $\langle e_2^*, e_3^*, e_4^*\rangle$ allows to assume $de_2^* = e_2^* \wedge e_3^*$. With this change the coefficients of the decomposition of de_1^* become arbitrary save for the conditions $A_1 = 0$ and $E_1 - B_1E_1 + C_1D_1 = 0$.

2.2.1. $B_1 = 1$, $C_1D_1 = 0$. We may assume $F_1 = 0$.

2.2.1.1. $C_1 = 0$, $E_1 \neq 0$. We get a special case of 2(i).

2.2.1.2. $C_1 = 0$, $E_1 = 0$. We get the 3-dimensional case considered above.

2.2.1.3. $D_1 = 0$, $E_1 \neq 0$. A transformation leads to case 2(i).

2.2.1.4. $D_1 = 0$, $E_1 = 0$, $C_1 \neq 0$. A transformation leads to 2(i), $\nu_1 = 0$, $\nu_2 = 1$.

2.2.1.5. $D_1 = 0$, $E_1 = 0$, $C_1 = 0$. We get the 3-dimensional case considered above.

2.2.2. $B_1 \neq 1$, $B_1 \neq 0$, $E_1 = \dfrac{C_1D_1}{B_1 - 1}$. We may assume $F_1 = 0$.

2.2.2.1. $C_1D_1 \neq 0$. We may assume $C_1 = B_1 - 1$. We get: $de_1^* = B_1e_1^* \wedge e_3^* + (B_1 - 1)e_1^* \wedge e_4^* + D_1e_2^* \wedge e_3^* + D_1e_2^* \wedge e_4^* = e_1^* \wedge e_3^* + D_1e_2^* \wedge (e_3^* + e_4^*) + (B_1 - 1)e_1^* \wedge (e_3^* + e_4^*)$, and after a transformation: $de_1^* = e_1^* \wedge e_3^* + e_2^* \wedge e_4^* + \nu e_1^* \wedge e_4^*$. May be lead to equations 2(i).

2.2.2.2. $C_1 \neq 0$, $D_1 = 0$. A transformation leads to 2(i), $\nu_1 = 0$, $\nu_2 = 1$.

2.2.2.3. $C_1 = 0$. We get the 3-dimensional case considered above.

2.2.3. $B_1 = 0$, $E_1 = -C_1D_1$.

2.2.3.1. $C_1D_1 \neq 0$. We may assume $C_1 = 1$, $F_1 = 0$. We get $de_1^* = e_1^* \wedge e_4^* + D_1e_2^* \wedge e_3^* - D_1e_2^* \wedge e_4^*$. Making the change $D_1(e_3^* - e_4^*) \mapsto e_3^*$ and after $e_3^* \leftrightarrow e_4^*$ we get equations 2(i).

2.2.3.2. $C_1 \neq 0$, $D_1 = 0$. We may assume $C_1 = 1$, $F_1 = 0$. A transformation leads to 2(i), $\nu_1 = 0$, $\nu_2 = 1$.

2.2.3.3. $C_1 = 0$. A transformation leads to 2(ii).

3. Jordan box with zero eigenvalue: $B_1 = 0$, $B_2 = 0$, $D_1 = \mu$, $D_2 = 0$. From the conditions ($\diamondsuit$) we have: $C_2\mu = 0$, $(C_1 - E_2)\mu = 0$.

3.1. $\mu = 0$. We have: $de_i^* = v_i^* \wedge e_4^*$, $v_i^* \in \langle e_1^*, e_2^*, e_3^* \rangle$, $i = 1, 2$.

3.1.1. $e_3^* \in \langle v_1^*, v_2^* \rangle$. In this case there exists a basis such that $de_1^* = v^* \wedge e_4^*$, $de_2^* = e_3^* \wedge e_4^*$, $v^* \in \langle e_1^*, e_2^* \rangle$. Up to changes there are two possibilities: $v^* = e_1^*$ and $v^* = e_2^*$. They lead to equations 2(ii) and 2(iii) correspondingly if use the transformation $e_4^* \mapsto e_3^*$, $e_3^* \mapsto -e_4^*$.

3.1.2. $v_i^* \in \langle e_1^*, e_2^* \rangle$. We get the 3-dimensional case considered above.

3.2. $\mu \neq 0$. Then $C_1 = E_2$, $C_2 = 0$.

3.2.1. $C_1 \neq 0$. We may assume $F_2 = 0$. After the transformation $e_3^* \leftrightarrow e_4^*$ we get the case considered in 2.2 above.

3.2.2. $C_1 = 0$. Then $F_2 \neq 0$. A transformation leads to $F_1 = 0$, and another one leads to $E_1 = 0$. We get equations 2(iii).

4. Pure imaginary conjugated roots: $B_1 = 0$, $B_2 = -1$, $D_1 = 1$, $D_2 = 0$. From the equations ($\diamondsuit$) we get: $C_1 = E_2$, $C_2 = -E_1$.

4.1. $C_1 \neq 0$. We may assume $C_1 = 1$. A transformation leads to $F_1 = 0$, and another leads to $C_2 = 0$. We get the equations: $de_1^* = e_1^* \wedge e_4^* + e_2^* \wedge e_3^*$, $de_2^* = -e_1^* \wedge e_3^* + e_2^* \wedge e_4^* + F_2 e_3^* \wedge e_4^*$. We may assume $F_2 = 0$. The transformation $e_3^* \leftrightarrow e_4^*$ leads to a special case of 2(i).

4.2. $C_1 = 0$. We may assume $F_1 = 0$, $C_2 = 0$. We get case 2(v).

5. Conjugated roots not belonging to coordinate axis: $B_1 = 1$, $B_2 = -a$, $D_1 = a$, $D_2 = 1$, $a \neq 0$. From the conditions ($\diamondsuit$) we have: $C_1 = E_2$, $C_2 = -E_1$. We may assume $F_1 = 0$. Since $a \neq 0$, we may assume $C_2 = 0$.

5.1. $C_1 \neq 0$. We may assume $C_1 = 1$. After the transformation $e_3^* + e_4^* \mapsto e_4^*$ we get the equations: $de_1^* = e_1^* \wedge e_4^* + a e_2^* \wedge e_3^*$, $de_2^* = -a e_1^* \wedge e_3^* + e_2^* \wedge e_4^* + F_2 e_3^* \wedge e_4^*$. We may assume $a = 1$, $F_2 = 0$. After the transformation $e_3^* \leftrightarrow e_4^*$ we get a special case of equations 3(i).

5.2. $C_1 = 0$. The equations take the form: $de_1^* = e_1^* \wedge e_3^* + a e_2^* \wedge e_3^*$, $de_2^* = -a e_1^* \wedge e_3^* + e_2^* \wedge e_3^* + F_2 e_3^* \wedge e_4^*$. One easily checks that in this case the Lie algebra is not symplectic.

3°. $\underline{\dim \mathcal{G}' = 1}$. In this case the only nonzero differential is degenerate. Actually if not then there exists a basis such that $de_1^* = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$, $de_2^* = de_3^* = de_4^* = 0$, from where $d^2 e_1^* \neq 0$. Thus, the differential de_1^* is decomposable and we get the 3-dimensional case considered above.

4°. $\underline{\dim \mathcal{G}' = 0}$. This is clearly the commutative case.

The statement on the uniqueness of the structures at the end of the theorem could be now easily checked. $\square$

Now when we have an exact symplectic algebra there may exist nonequivalent symplectic structures. Modulo closed 1-forms the set of exact structures is parametrized by covectors whose differentials are nondegenerate. These covectors among all covectors are organized in orbits of the coadjoint action.

Lemma 2. *If a covector $f^* \in \mathcal{G}^*$ is such that df^* is nondegenerate then the orbit of coadjoint action through it is open. In other words for any sufficiently close covector $\hat{f}^*$ there exists an automorphism of the Lie algebra such that it sends f^* to $\hat{f}^*$.*

Proof. The tangent space to the orbit through f^* is $T_{f^*}\mathcal{O} = \langle ad^*_X(f^*) \,|\, X \in \mathcal{G}\rangle$. Its annulator is the linear space of $Y \in \mathcal{G}$ such that $\langle ad^*_X(f^*), Y\rangle = 0$ for every $X \in \mathcal{G}$. This means that $\langle f^*, [X, Y]\rangle = 0$ or $df^*(X, Y) = 0$ for all X which yields $Y = 0$ and $T_{f^*}\mathcal{O} = \mathcal{G}^*$. $\square$

2. Elliptic Monge-Ampère equations

According to the paper [7] Monge-Ampère equations may be considered as half-dimensional effective forms on the elements of some contact distribution (it's better to call such an object the generalized Monge-Ampère equation). The pointwise classification (i.e. with respect to the linear symplectic group at a point) of such equations was given in [8]. We consider only the case of Monge-Ampère with two independent variables. This means that we consider a four-dimensional symplectic manifold (M^4, ω) and a 2-form $\theta \in \Omega^2(M)$, $\theta \wedge \omega = 0$.

2.1. Classification of elliptic equations depending on two variables

For the elliptic Monge-Ampère equations with two variables there exists an alternative description. Let normalize θ by the condition $\mathrm{Pf}(\theta) = 1$ and define an automorphism j by the formula $i_j\omega = \theta$ where $i_\bullet$ is the substitution. Then j is an almost complex structure. The Monge-Ampère equation (ω, j)

is called nondegenerate if j is of general position (at least it is nonintegrable and the equation is not of divergence type). For these equations the paper [6] contains the following classificational result:

Theorem 4. *A nondegenerate elliptic Monge-Ampère equation (ω, j) canonically determines an $\{e\}$-structure, i.e. the field of basis frames (P_1, P_2, Q_1, Q_2). This structure is a complete invariant, i.e. two nondegenerate elliptic Monge-Ampère equations are isomorphic if and only if the corresponding $\{e\}$-structures are. The classifying $\{e\}$-structure satisfies the following relations which completely determine it (we use the dual basis in the cotangent space):*

$$\omega = P_1^* \wedge Q_1^* + P_2^* \wedge Q_2^*, \quad \theta = P_1^* \wedge Q_2^* - P_2^* \wedge Q_1^*,$$

$$j = P_1^* \otimes P_2 - P_2^* \otimes P_1 - Q_1^* \otimes Q_2 + Q_2^* \otimes Q_1,$$

$$N_j = -P_1^* \wedge Q_1^* \otimes P_2 + P_1^* \wedge Q_2^* \otimes P_1 - P_2^* \wedge Q_1^* \otimes P_1 - P_2^* \wedge Q_2^* \otimes P_2.$$

Here $N_j = [j, j]$ is the Nijenhuis self-bracket of the almost complex structure j. Note that since for $\{e\}$-structures the equivalence problem is solved (see [9]) this theorem serves as equivalence criterion for two-dimensional elliptic nondegenerate Monge-Ampère equations.

Corollary. *Every symmetry of a Monge-Ampère equation is a symmetry for its $\{e\}$-structure invariant and vice versa.* □

Thus we are given a tool for symmetries constructing for the equations of the described type. Moreover this is the key idea for considering the structures of the next section.

2.2. Lie group action and invariant equations

It is natural to consider equivariant equations, i.e. the equations with transitive action of some Lie group. We assume that the elements of this group depend on the first derivatives of the solutions. Let us call equations of these type *invariant equations*.

In our model we must permit a Lie group action on the symplectic manifold (M^4, ω) which preserves the structures ω and j. Since the action is transitive our manifold is homogeneous. Let us note that according to the canonicity of formulas from theorem 4 this is equivalent to the invariance of the classifying $\{e\}$-structure. Thus the structural functions c^k_{ij} for $\{e\}$-structure e_i, $[e_i, e_j] = c^k_{ij} e_k$, are constant and our manifold becomes a Lie group. As in section 1 we may assume everything to live on the corresponding Lie algebra. For example the Nijenhuis tensor

$$N_j(\xi, \eta) = [j\xi, j\eta] - j[j\xi, \eta] - j[\xi, j\eta] - [\xi, \eta]$$

can be calculated by means of the Lie algebra commutators.

Theorem 4 now also takes place and the construction of the $\{e\}$-structure from [6] may be re-written for the invariant situation in the interior Lie algebra terms.

Note that the $\{e\}$-structure appeared is not arbitrary. There are two differential conditions on it. First the form ω is closed. Setting $e_1 = P_1$, $e_2 = P_2$, $e_3 = Q_1$, $e_4 = Q_2$ from the structural equations above wee obtain:

$$c^1_{12} - c^3_{23} + c^4_{13} = c^2_{12} - c^3_{24} + c^4_{14} = c^1_{14} - c^2_{13} + c^3_{34} = c^1_{24} - c^2_{23} + c^4_{34} = 0.$$

The second condition is connected with the almost complex structure. We may define this structure j by means of formulas of theorem 4 and then we compute the Nijenhuis tensor. The condition is that it coincides with the tensor N_j given in theorem 4. If these two conditions hold true we may recover the Monge-Ampère equation.

2.3. Example of the recovering a Monge-Ampère equation by its $\{e\}$-structure

Let two necessary conditions discussed in 2.2 be satisfied. Define coordinates in a neighborhood of the unity basing on the exponential mapping $\exp : \mathcal{G} \to G$. Campbell-Hausdorff formula shows how to write left-invariant vector fields in these coordinates ([5], [10]). This allows us to write down the generalized Monge-Ampère equation. To obtain the ordinary one ([7]) we need to fix a Lagrangian submanifold $L^2 \subset M^4$ (or a local cotangent bundle with θ-trivial Lagrangian fibers L^2 in order to obtain a quazilinear equation), identify its neighborhood with cotangent bundle $UL \simeq T^*L$ and substitute the expression $p = \partial u/\partial q$ into the equation $\theta(p, q) = 0$ in canonical coordinates (p, q). The last operation may result in different Monge-Ampère for different choices of L (however for nonequivalent admissible $\{e\}$-structures or for nonequivalent generalized Monge-Ampère equations all possible as representative ordinary Monge-Ampères are different).

Let's demonstrate the scheme for a nilpotent Lie algebra (we may call the obtained equation nilpotent). A nilpotent Lie algebra on $\mathbb{R}^4$ is isomorphic to one of the cases: commutative, 3-dimensional or 4-dimensional. Let us consider the last case. It is determined by the relations: $[e_2, e_3] = e_1$, $[e_3, e_4] = e_2$ and $[e_i, e_j] = 0$ for all the others $i < j$. Let us consider the left-invariant basis of the Lie algebra of the form: $P_1 = e_1$, $P_2 = e_2$, $Q_1 = e_3$, $Q_2 = e_4$. This basis satisfies two necessary conditions from the end of 2.2. Write it in exponential coordinates. Note that for 4-dimensional nilpotent case the series in the Campbell-Hausdorff formula terminates on the third term. Thus the left-invariant vector field through a point $x \in \mathcal{G}$ taking the value $\eta \in \mathcal{G}$ at zero has the form (L_x denotes the left shift):

$$(L_x)_*\eta = \eta + \frac{1}{2}[x, \eta] + \frac{1}{12}[x, [x, \eta]].$$

Extending left-invariantly the basis $e_i = \partial_i = \dfrac{\partial}{\partial x_i}$ we get the expression for the basis in coordinates x:

$$\begin{array}{ll} P_1 = \partial_1, & Q_1 = \partial_3 + (\frac{1}{2}x_2 + \frac{1}{12}x_3x_4)\partial_1 - \frac{1}{2}x_4\partial_2, \\ P_2 = \partial_2 - \frac{1}{2}x_3\partial_1, & Q_2 = \partial_4 - \frac{1}{12}(x_3)^2\partial_1 + \frac{1}{2}x_3\partial_2. \end{array}$$

For the dual basis we have the expression:

$$\begin{array}{ll} P_1^* = dx_1 + \frac{1}{2}x_3dx_2 - (\frac{1}{2}x_2 - \frac{1}{6}x_3x_4)dx_3 - \frac{1}{6}(x_3)^2dx_4, & Q_1^* = dx_3, \\ P_2^* = dx_2 + \frac{1}{2}x_4dx_3 - \frac{1}{2}x_3dx_4, & Q_2^* = dx_4. \end{array}$$

Note that the last expressions may be obtained from the Maurer-Cartan equations $de_1^* = e_3^* \wedge e_2^*$, $de_2^* = e_4^* \wedge e_3^*$, $de_3^* = de_4^* = 0$. Further:

$$\begin{aligned} \omega &= P_1^* \wedge Q_1^* + P_2^* \wedge Q_2^* \\ &= dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + \frac{1}{2}x_3dx_2 \wedge dx_3 \\ &\quad +(\frac{1}{2}x_4 + \frac{1}{6}(x_3)^2)dx_3 \wedge dx_4. \end{aligned}$$

$$
\begin{aligned}
\theta &= P_1^* \wedge Q_2^* - P_2^* \wedge Q_1^* \\
&= dx_1 \wedge dx_4 - dx_2 \wedge dx_3 + \frac{1}{2}x_3 dx_2 \wedge dx_4 \\
&\quad +(-\frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{6}x_3x_4)dx_3 \wedge dx_4.
\end{aligned}
$$

Let change coordinates:

$$
\begin{aligned}
p_1 &= x_1 + \tfrac{1}{2}x_4 + \tfrac{1}{4}x_3^2 + \tfrac{1}{2}x_2x_3 - \tfrac{1}{2}x_4^2 - \tfrac{1}{6}x_3^2x_4, \quad && q_1 = x_3, \\
p_2 &= x_2 + \tfrac{1}{2}x_3 - \tfrac{1}{2}x_3x_4, && q_2 = x_4.
\end{aligned}
$$

This choice is equivalent to the choice of the general solution $L^2 = \{q_1 = c_1; q_2 = c_2\}$, i.e. such a Lagrangian manifold that $\theta|_L \equiv 0$. In new coordinates we have:

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

$$\theta = dp_1 \wedge dq_2 - dp_2 \wedge dq_1 - p_2 dq_1 \wedge dq_2.$$

Substitute the expressions $p_1 = \partial u/\partial q_1$, $p_2 = \partial u/\partial q_2$ to the equation $\theta = 0$ and we get the Monge-Ampère equation which we looked for:

$$\triangle u = \frac{\partial u}{\partial q_2}.$$

2.4. Scheme of normal forms for Monge-Ampères on two-dimensional surfaces

Now since theorem 4 gives an equivalence criterion for nondegenerate elliptic Monge-Ampère equations with two variables it's possible to determine normal forms of these equations. We suppose that the situation is equivariant and so we have a Lie group with invariant symplectic and almost complex structures. As usual we talk instead of Lie algebras.

We may use normal forms of symplectic structures on Lie algebras as in 1.4. Then we add an almost complex structure j to the pair $(\mathcal{G}, \omega)$ with $\mathcal{G}$ a Lie algebra and ω a symplectic form. Now this form is not arbitrary. It must satisfy the condition $\omega(X, jX) = 0$. Denote by $\mathcal{J}$ the set of all complex structures on a linear space V and by $\mathcal{J}^{\pm}(V, \omega)$ the set

$\{j \in \mathcal{J}(V) \,|\, \omega(jX, jY) = \pm\omega(X, Y)\}$. The set $\mathcal{J}^+(V, \omega)$ is not even connected (contrary to the set $\mathcal{J}^+(V, \omega) \cap \{j \in \mathcal{J} \,|\, \omega(X, jX) > 0,\ X \neq 0\}$ which as well-known in symplectic geometry is contractible). We are interested in the set $\mathcal{J}^-(V^4, \omega)$. This manifold is diffeomorphic to $S^2 \times \mathbb{R}^2$. Actually instead of j we may consider 2-forms $\theta(X, Y) = \omega(jX, Y)$ which satisfy the conditions $\theta \wedge \omega = 0$, $\langle \theta, \theta \rangle = 1$. Here $\langle \theta_1, \theta_2 \rangle = \dfrac{\theta_1 \wedge \theta_2}{\omega \wedge \omega}$ is the metric associated with the Pfaffian. This metric has type $(3;3)$. The orthogonal complement to ω has type $(2;3)$ and now the equality $\mathcal{J}^-(\mathcal{G}^4, \omega) \simeq S^2 \times \mathbb{R}^2$ is clear. Note that actually this gives an obstruction to existence of a nonvanishing generalized elliptic Monge-Ampère equation on a symplectic manifold (M^4, ω), this is the first obstruction for constructing a section to the bundle $\mathcal{J}^-(T_x M^4; \omega_x) \mapsto x$ and it lies in $H^3(M^4; \mathbb{Z})$. However in the case of Lie groups this obstruction is always zero. Actually there always exists a left-invariant section.

Thus to the normal symplectic forms from 1.4 we can add the structures from $\mathcal{J}^-(\mathcal{G}^4; \omega)$ and this would give us through the recovering procedure of 2.3 the normal forms of elliptic Monge-Ampère equations with two variables.

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