Geometrical structures on solution spaces of Ordinary Differential Equations

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Contents

Abstract

In this text we will introduce the notion of solution space of a integrable distribution on a manifold, as a quotient space of this manifold, and investigate criteria for tensors to be invariant on our distribution. This will provide the algebraic framework in which we will place Cartan distributions of differential equations. We shall look at ODEs of order k (that can be resolved with respect to the highest derivative), and especially the case $k = 2$.
For $k = 2$ we shall see examples of equations that possess (co-)symmetries, metric structure and symplectic structure, all in the sense of invariant tensors.

Chapter 1

The solution space of a distribution

1.1 Involutive distributions

Let P be a distribution of dimension n and codimension k on a smooth manifold M. That is, $dim M = n + k$ and at each $x \in M$ an n-dimensional subspace $P_x \subset T_x M$ is specified such that P_x varies smoothly with x. We define

$$\Delta(P) = \{X \in \mathcal{D}(M) \mid X_m \in P_m \ , \ \forall m \in M\}$$

$$Ann\,(P) = \{\omega \in \Omega^1(M) \mid \omega(X) = 0 \ , \ \forall X \in \Delta(P)\ \}$$

These are both $C^\infty(M)$-modules, always locally free. Our distribution is fixed whenever one specifies a (local) basis of either.

Definition 1.1 *A submanifold $N \subset M$ is an integral manifold of the distribution P if $T_x N \subset P_x$ for every $x \in N$. P is said to be completely integrable if $\forall x \in M$, $\exists$ integral manifold N through x such that* $\dim N = n$.

Let $\{X_i\}_{i=1}^n$ be a (local) basis of $\Delta(P)$. We say that the distribution P is in **involution** if around any point of M there exists a local basis $\{X_i\}_{i=1}^n$ such that

$$[X_i, X_j] = \sum_{r=1}^{n} c_{ij}^r X_r$$

for some $c_{ij}^r \in C^\infty(M)$.

Theorem 1.1 *(Frobenius theorem)*
P is completely integrable if and only if it is in involution.

A proof of this theorem can be found in [Boo75].

1.2 The solution space of a distribution P

Assume we have a completely integrable distribution P on a smooth manifold M. P beeing completely integrable ensures us that through each $x \in M$ there passes a connected integral manifold $N = N(x)$ of P of dimension n. This gives us the following equivalence-relation defined on our manifold M :

$$Given\ x, y \in M\ ,\ we\ say\ that\ x \sim y\ ,\ if\ y \in N = N(x)$$

This relation is easily checked to be an equivalence relation. Given distribution P as above, we shall call

$$\mathcal{S} = M/\sim$$

the space of solutions of P. Points in this quotientspace correspond to connected integral manifolds of our distribution. We will try to define functions and tensors on this "invisible" solution space in terms of M. The main idea is to consider objects over M which are invariant on our distribution, in the sense defined below. For a distribution on a manifold we have the notion of a symmetry. Let $\{A_t\}$ be the flow generated by $X \in \mathcal{D}(M)$ on M. X is a **symmetry** of the distribution P if for any t, and any $x \in M$, $(A_t)_*(P_x) \subset P_{A_t x}$ We denote the symmetries of P by $Sym(P)$.

Theorem 1.2 *The following are equivalent :*

(1) $X \in Sym(P)$

(2) $L_X(\Delta(P)) = [X, \Delta(P)] \subset \Delta(P)$

(3) $L_X(\,Ann(P)\,) \subset Ann(P)$

For proof of the theorem, see [DL91].

Figure 1.1 *Distribution P on manifold M*

Our distribution P is in involution, so for all $1 \leq i, j \leq n$ we have $[X_i, X_j] = \sum_{r=1}^{n} c_{ij}^r X_r$, where $\{X_i\}_{i=1}^n$ is an appropriate local basis of $\Delta(P)$. Thus $\Delta(P) \subset Sym(P)$, and we define

$$Char(P) = \Delta(P) \cap Sym(P)$$

the **characteristic symmetries** of P.
Let

$$Shuf(P) = Sym(P)/Char(P)$$

denote the **shuffling symmetries** of P. The characteristic symmetries of P leave each maximal integral manifold, whereas the shuffling symmetries will move one into another, hence the name. If our space $\mathcal{S}$ possesses manifold structure, we can explain geometrically how to interpret a symmetry on M as a vector field on $\mathcal{S}$. But before we look into cases when $\mathcal{S} \in Sman$ we will make some general constructions.

1.3 Functions and vectorfields on solution spaces

Definition 1.2 *We define*

$$\mathcal{F}(P) = \{f \in C^\infty(M) \mid X \lrcorner\, df = 0 \ , \ \forall X \in \Delta(P)\}$$

to be the algebra of 1st-integrals of P.

The requirement $X \lrcorner\, df = df(X) = X(f) = L_X(f) = 0$ gives us functions that are **constant** on integral manifolds of P, so-called **1st-integrals** of our distribution. Any such f will give a well-defined function on $\mathcal{S}$, and $\mathcal{F}(P)$ is precisely the set $\{ f \in C^\infty(M) \,|\, f(x) = f(y) \; \forall \; x \sim y \}$.

Proposition 1.1 *$\mathcal{F}(P)$ is an $\mathbb{R}$-algebra with respect to usual addition and multiplication of functions, and multiplication by reals.*

Proof : We need only check that for all $f, g \in \mathcal{F}(P)$, $f + g$ *and* $fg \in \mathcal{F}(P)$.
We have
$L_X(f+g) = L_X(f) + L_X(g) = 0$ and
$L_X(fg) = fL_X(g) + gL_X(f) = 0$
The rest of the properties are obvious. □

We want to define the objects that should correspond to vector fields on $\mathcal{S}$. They should be derivations of $\mathcal{F}(P)$, and really give meaning as vector fields in the cases when $\mathcal{S} \in Sman$.

Definition 1.3 *Given P as above we define*

$$\mathcal{D}(P) = Shuf(P)$$

to be the Lie-algebra of **vector fields** *of our distribution P.*

Strictly speaking, the elements of $\mathcal{D}(P)$ are *classes* of vector fields modulo the characteristic vector fields. Two representatives Y and Y' of a class in $\mathcal{D}(P)$ differ by an element $X \in \Delta(P)$, $Y' = Y + X$. The flows generated by Y and Y' respectively will move an integral manifold of P into exactly the same new integral manifold. Thus two representatives of the same class in $\mathcal{D}(P)$ will move a point $[x]$ in $\mathcal{S}$ to the same point $[y] \in \mathcal{S}$, so it is well-defined to consider this the action of the class of Y.

$\mathcal{D}(P)$ inherits the bracket of $Sym(P)$ in the following way

$$\left[\overline{X}, \overline{Y}\right] \stackrel{\text{def}}{=} \overline{[X,Y]}$$

For any $X_0 \in \Delta(P)$, $X, Y \in Sym(P)$ we get $\overline{[X + X_0, Y]} = \overline{[X,Y]}$ since $[X_0, Y] \in \Delta(P)$. Hence, the bracket operation on $\mathcal{D}(P)$ is well defined. It inherits the Lie-algebra properties of the bracket on $Sym(P)$.

Moreover, we can interpret the elements of $\mathcal{D}(P)$ as **derivations** of $\mathcal{F}(P)$.

Definition 1.4 *Given $Y \in Sym(P)$, a representative of an element $\overline{Y} \in \mathcal{D}(P)$, we define the operator*

$$\overline{Y} : \mathcal{F}(P) \longrightarrow \mathcal{F}(P)$$

by taking $\overline{Y}(f) = Y(f)$.

Let $Y' = Y + X$ be another representative of $\overline{Y}$, $X \in \Delta(P)$. Then $Y'(f) = Y(f) + X(f) = Y(f)$, so the operator is well-defined. The operator inherits the derivational properties of Y, and the only thing we have to check, is that $Y(f) \in \mathcal{F}(P)$ for all $f \in \mathcal{F}(P)$. But

$$L_X(\, Y(f)\,) = X(Y(f)) = [X, Y](f) - Y(X(f)) = 0$$

since $X, [X, Y] \in \Delta(P)$, so $Y(f) \in \mathcal{F}(P)$.
Also, $\mathcal{D}(P)$ is clearly a $\mathcal{F}(P)$-module.

1.4 The algebra of differential forms on solution spaces

Differential 1-forms on our space $\mathcal{S}$ should be objects dual to vectorfields $\mathcal{D}(P)$. Thus we make the following step:

Definition 1.5 *We define*

$$\Omega^1(P) = \left\{ \omega \in \Omega^1(M) \;\middle|\; \begin{array}{ll} X \lrcorner\, \omega = 0 & \textit{and} \\ X \lrcorner\, d\omega = 0 & \forall X \in \Delta(P) \end{array} \right\}$$

This ensures us that $L_X(\omega) = 0$ for $X \in \Delta(P)$, and that $\langle \omega, Y \rangle \in \mathcal{F}(P)$ where Y is a representative of a class in $\mathcal{D}(P)$. We check the latter:

$$L_X \langle \omega, Y \rangle = \langle L_X(\omega), Y \rangle + \langle \omega, L_X(Y) \rangle = 0$$

since $L_X(\omega) = 0$ and $L_X(Y) \in \Delta(P)$. This gives us a way to find 1st-integrals of our distribution, provided we know elements of $\mathcal{D}(P)$ and $\Omega^1(P)$. Moreover, we can make the following definition.

Definition 1.6 *We define the* **pairing**

$$\langle\ ,\ \rangle\ :\ \Omega^1(P)\times\mathcal{D}(P)\longrightarrow\mathcal{F}(P)$$

by taking $\left\langle \omega,\overline{X}\right\rangle \stackrel{\text{def}}{=} \langle\omega,X\rangle=\omega(X)$, *where* $X\in Sym(P)$ *is any representative of* $\overline{X}\in\mathcal{D}(P)$, *and* $\omega\in\Omega^1(P)$.

It is easy to see that this is well-defined, assume $\overline{X'}=\overline{X}$ in $\mathcal{D}(P)$, then $X'=X+X_0$ where $X_0\in\Delta(P)$. This gives

$$\omega(X')=\omega(X+X_0)=\omega(X)+\omega(X_0)=\omega(X)$$

The pairing is $\mathcal{F}(P)$**-bilinear**

$$\left\langle f\omega,\overline{X}\right\rangle=\left\langle\omega,f\overline{X}\right\rangle=f\left\langle\omega,\overline{X}\right\rangle$$

for any $f\in\mathcal{F}(P)$. Thus it plays the same role as the usual pairing of vectorfields and differentialforms.

We shall follow this line of construction further, and find out what corresponds to l-forms, differentials and symmetric 2-forms in our picture.

Definition 1.7 *We define*

$$\Omega^l(P)=\left\{\omega\in\Omega^l(M)\ \middle|\ \begin{array}{ll} X\ \lrcorner\ \omega=0 & and \\ X\ \lrcorner\ d\omega=0 & \forall X\in\Delta(P)\end{array}\right\}$$

and the operator

$$\tilde{d}:\Omega^l(P)\longrightarrow\Omega^{l+1}(P)\qquad l\geq 0$$

by taking $\tilde{d}(\omega)=d\omega$, $\omega\in\Omega^l(P)$.
We define $\Omega^0(P)=\mathcal{F}(P)$, *and* $\Omega^l(P)=0$ *when* $l<0$.

Remark :
We have to show that $\tilde{d}$ really maps $\Omega^l(P)\subset\Omega^l(M)$ into $\Omega^{l+1}(P)$. Let $\omega\in\Omega^l(P)$, $X\in\Delta(P)$. Then

$$X\ \lrcorner\ \tilde{d}(\omega)=X\ \lrcorner\ d\omega=0\quad and\quad \tilde{d}(d\omega)=d^2\omega=0$$

so $\tilde{d}\omega=d\omega\in\Omega^{l+1}(P)$. Moreover, $\tilde{d}$ inherits the properties of the usual differential.

Proposition 1.2 *Let P be an involutive distribution of dim$= n$, codim$= k$. Then locally any element $\theta \in \Omega^l(P)$ is on the form*

$$\theta = \sum \alpha_{(i_1 \ldots i_l)} \theta_{i_1} \wedge \ldots \wedge \theta_{i_l}$$

where the θ_{i_j}-s form a local basis of $Ann(P)$

Proof :
Let $\{\theta_i\}_{i=1}^{k+n}$ be a local basis of $\Omega^1(M)$ such that $\{\theta_i\}_{i=1}^{k}$ generates $Ann(P)$. Take $\{E_j\}_{j=1}^{k+n}$ to be the local basis of $\mathcal{D}(M)$ such that $\{E_j\}_{j=k+1}^{k+n}$ generates $\Delta(P)$ and $\theta_i(E_j) = \delta_{ij}$. Any element θ of $\Omega^l(P) \subset \Omega^l(M)$ is on the form

$$\theta = \sum_{1 \leq i_1 < \ldots < i_l \leq k+n} \alpha_\sigma \theta_{i_1} \wedge \ldots \wedge \theta_{i_l}$$

for some $\alpha_\sigma \in C^\infty(M)$, where $\sigma = (i_1, \ldots, i_l)$. By definition of $\Omega^l(P)$ we have that $E_s \lrcorner\ \theta = 0$ for $s = k+1, \ldots, k+n$. By calculation we have that

$$E_s \lrcorner\ \theta_{i_1} \wedge \ldots \wedge \theta_{i_l} = \begin{cases} (-1)^{r-1} \theta_{i_1} \wedge .. \wedge \theta_{i_{r-1}} \wedge \theta_{i_{r+1}} \wedge .. \wedge \theta_{i_l} & \text{if } s = i_r \\ 0 & \text{otherwise} \end{cases}$$

Thus for $k+1 \leq s \leq k+n$ we have

$$E_s \lrcorner\ \theta = \sum_{\substack{1 \leq i_1 < i_2 < \ldots < i_l \\ i_l = k+1}}^{k+n} \alpha_\sigma E_s \lrcorner\ \theta_{i_1} \wedge \ldots \wedge \theta_{i_l}$$

By taking $s = k+n$, we get that

$$\sum_{1 \leq i_1 < \ldots < i_{l-1} < k+n} (-1)^{l-1} \alpha_{(i_1, .., i_{l-1}, k+n)} \theta_{i_1} \wedge \ldots \wedge \theta_{i_{l-1}} = 0$$

This implies that $\alpha_{(i_1, .., i_{l-1}, k+n)} = 0$ for all $1 \leq i_1 < \ldots < i_{l-1} \leq k+n-1$. Repeating this for $s = k+n-1$ and then consequently to $s = k+1$ we get that $\alpha_{(i_1, .., i_l)} = 0$ whenever $i_l > k$. Thus

$$\theta = \sum_\sigma \alpha_\sigma \theta_{i_1} \wedge \ldots \wedge \theta_{i_l}$$

for $\alpha_\sigma \in C^\infty(M)$ and $\theta_{i_j} \in Ann(P)$. $\square$

Remark: We only used part (i) of the requirements on elements of $\theta \in \Omega^l(P)$, namely that $\Delta \lrcorner\ \theta = 0$. Combining this with the requirement $L_\Delta \theta = 0$ will give the conditions on the functions α_σ. These calculations will be done for the 1-forms $\Omega^1(\mathcal{C}_\mathcal{E})$ of a special distribution $\mathcal{C}_\mathcal{E}$ in **Section 3.2**.

Corollary 1.1

$$\Omega^l(P) = 0 \quad for \quad l > k$$

This follows immediately from **Proposition 1.2**.

To sum up the algebraic properties of our constructions in this section, we state the following theorem.

Theorem 1.3 *Given P as above*

$$\Omega^{\cdot}(P) = \bigoplus_{s\in\mathbb{Z}} \Omega^s(P)$$

is a $\mathbb{Z}$-graded σ-commutative algebra w.r.t. the usual wedge product

$$\wedge : \Omega^s(P) \times \Omega^t(P) \longrightarrow \Omega^{s+t}(P)$$

σ-commutativity means that $\omega \wedge \theta = \sigma(s,t)(\theta \wedge \omega)$ where $\sigma(s,t) = (-1)^{st}$. Also, the operator $\tilde{d}$ is a derivation of degree $+1$ of $\Omega^{\cdot}(P)$

Proof :
We need to prove that $\theta \wedge \omega \in \Omega^{s+t}(P)$ for any $\theta \in \Omega^s(P)$, $\omega \in \Omega^t(P)$. Let $X \in \Delta(P)$. Then

$$X \lrcorner (\theta \wedge \omega) = (X \lrcorner \theta) \wedge \omega + (-1)^s \theta \wedge (X \lrcorner \omega) = 0$$

Moreover, $d(\theta \wedge \omega) = d\theta \wedge \omega + (-1)^s \theta \wedge d\omega$, and by a calculation similar to the one above we get

$$X \lrcorner d(\theta \wedge \omega) = 0$$

which implies that $\theta \wedge \omega \in \Omega^{s+t}(P)$.
Each $\Omega^s(P)$ is a $\mathcal{F}(P)$-module :

$$X \lrcorner f\omega = fX \lrcorner \omega = 0$$

for every $X \in \Delta(P), \omega \in \Omega^s(P), f \in \mathcal{F}(P)$, and

$$X \lrcorner \tilde{d}(f\omega) = X \lrcorner (df \wedge \omega) = 0$$

since $df \wedge \omega \in \Omega^{s+1}(P)$. Thus $f\omega \in \Omega^s(P)$.
It is obvious that $\Omega^s(P)$ is closed under usual addition of forms.
The properties of $\wedge$ and $\tilde{d}$ are clear. □

1.5 Further properties of $\Omega^l(P)$

We wish to see if $\Omega^l(P)$ really fits into the algebraic picture of differential-forms versus vectorfields $\mathcal{D}(P)$, and functions $\mathcal{F}(P)$.
We shall see that the elements of $\Omega^l(P)$ algebraically play the role of l-forms.

Definition 1.8 *Given an element $\theta \in \Omega^l(P)$ we define the following map:*

$$\tilde{\theta} : \underbrace{\mathcal{D}(P) \times \ldots \times \mathcal{D}(P)}_{l} \longrightarrow \mathcal{F}(P)$$

by taking

$$\tilde{\theta}(\overline{X}_1, \ldots, \overline{X}_l) \stackrel{\text{def}}{=} \theta(X_1, \ldots, X_l)$$

where $X_i \in Sym(P)$ are any representatives of elements $\overline{X}_i \in \mathcal{D}(P)$.

We will sometimes use the notations

$$\tilde{\theta}(\overline{X}_1, \ldots, \overline{X}_l) = \left\langle \tilde{\theta}; \overline{X}_1, \ldots, \overline{X}_l \right\rangle = \langle \theta; X_1, \ldots, X_l \rangle$$

and not distinguish between θ and $\tilde{\theta}$. To see that this is a well-defined operation we have to check independence of choice of representatives in $Sym(P)$. Assume $\overline{X'}_j = \overline{X}_j$ for some $1 \leq j \leq l$. Then $X'_j = X_j + X_0$ for some element $X_0 \in \Delta(P)$. It follows that

$$\theta(X_1, ., X'_j, ., X_l) =$$

$$= \theta(X_1, ., X_j, ., X_l) + (-1)^j \left(X_0 \lrcorner\ \theta\right) (X_1, ., X_{j-1}, X_{j+1}, ., X_l) =$$
$$= \theta(X_1, ., X_j, ., X_l)$$

since $X_0 \lrcorner\ \theta = 0$ by the definition of $\Omega^l(P)$. Thus $\tilde{\theta}$ is well-defined and clearly inherits the tensorproperties; it is an l-alternating, $\mathcal{F}(P)$-additive operator. What is left to check, is that $\theta(X_1, \ldots, X_l)$ is really an element of $\mathcal{F}(P)$ for any $X_i \in Sym(P), \theta \in \Omega^l(P)$. Given $X \in \Delta(P)$, we make the following calculation:

$$L_X\left(\theta(X_1, \ldots, X_l)\right) = L_X \circ i_{X_k} \circ \ldots \circ i_{X_1}(\,\theta\,) =$$

$$\left[i_{X_k} \circ L_{X_k} \circ \ldots \circ i_{X_1} \;+\; i_{[X, X_k]} \circ i_{X_{k-1}} \circ \ldots \circ i_{X_1} \right](\,\theta\,)$$

$$= i_{X_k} \circ L_{X_k} \circ \ldots \circ i_{X_1}(\,\theta\,) = \ldots$$

$$\ldots = i_{X_k} \circ \ldots \circ i_{X_1} \circ L_X(\,\theta\,) = 0$$

since $L_X \circ i_{X_i} - i_{X_i} \circ L_X = i_{[X,X_i]}$ for all X_i, and in addition $i_{X_k} \circ \ldots \circ i_{[X,X_i]} \circ \ldots \circ i_{X_1}(\,\theta\,) = 0$ since $[X, X_i] \in \Delta(P)$ for all $i = 1, .., k$.

Let us consider tensorproducts of $\Omega^1(P)$ over $\mathcal{F}(P)$. A general element θ of $(\Omega^1(P))^{\otimes l}$ is on the form $\theta = \sum \alpha_{i_1..i_l}\theta_{i_1} \otimes \cdot \cdot \otimes \theta_{i_l}$ where $\theta_{i_j} \in \Omega^1(P)$ and $\alpha_{i_1..i_l} \in \mathcal{F}(P)$.

Definition 1.9 *Let $\theta \in (\Omega^1(P))^{\otimes l}$. We define*

$$\tilde{\theta} : \underbrace{\mathcal{D}(P) \times \ldots \times \mathcal{D}(P)}_{l} \longrightarrow \mathcal{F}(P)$$

by taking

$$\tilde{\theta}\left(\overline{X}_1, ..., \overline{X}_l\right) \stackrel{\text{def}}{=} \sum \alpha_{i_1..i_l}\theta_{i_1}(X_1) \cdots \theta_{i_l}(X_l)$$

for any representatives X_i of $\overline{X}_i \in \mathcal{D}(P)$.

It follows from our earlier investigations that this is well-defined with respect to choice of representatives X_i. To see that $\tilde{\theta}\left(\overline{X}_1, ..., \overline{X}_l\right)$ is an element in $\mathcal{F}(P)$, we need only observe that $\theta_{i_j}(X_i) \in \mathcal{F}(P)$ for $i = 1..l$, and thus this is true also for the sum $\sum \alpha_{i_1..i_l}\theta_{i_1}(X_1) \cdots \theta_{i_l}(X_l)$. $\tilde{\theta}$ inherits the tensorproperties of θ.

We note that the anti-symmetrisation operator

$$\Lambda : \left(\Omega^1(P)\right)^{\otimes l} \longrightarrow \Omega^l(P)$$

defined by

$$\Lambda\left(\sum \alpha_{i_1...i_l}\theta_{i_1} \otimes \cdot \cdot \otimes \theta_{i_l}\right) = \sum \alpha_{i_1...i_l}\theta_{i_1} \wedge \ldots \wedge \theta_{i_l}$$

for $\alpha_{i_1...i_l} \in \mathcal{F}(P)$, $\theta_{i_j} \in \Omega^1(P)$ will map $(\Omega^1(P))^{\otimes l}$ into $\Omega^l(P)$. That is, wedge products of elements in $\Omega^1(P)$ with coefficients in $\mathcal{F}(P)$, and their sums, will be elements of $\Omega^l(P)$. This follows from **Theorem 1.3**.

1.6 Morphisms of distributions

For manifolds the natural map between them are smooth maps. Any smooth map $\phi : M \longrightarrow N$ between smooth manifolds M and N induces the following homomorphism of tangentspaces for every $a \in M$

$$\phi_{*,a} : T_a M \longrightarrow T_{\phi(a)} N$$

called the **differential of ϕ at a**. It is defined by

$$\phi_{*,a}(X_a)(f) = X_a(f \circ \phi)$$

for any $X \in \mathcal{D}(M)$, $f \in C^\infty(N)$. We will denote the dual linear map of $\phi_{*,a}$ by

$$\phi^*_{\phi(a)} : T^*_{\phi(a)} N \longrightarrow T^*_a M$$

Given $\omega \in \Omega^1(N)$, $X \in \mathcal{D}(M)$ and $a \in M$ we have

$$\Big\langle \phi^*_{\phi(a)}(\,\omega_{\phi(a)}\,), X_a \Big\rangle \stackrel{def}{=} \Big\langle \omega_{\phi(a)}, \phi_{*,a}(X_a) \Big\rangle$$

This formula determines a smooth 1-form $\phi^*(\omega) \in \Omega^1(M)$, thus we have the induced homomorphism

$$\phi^* : \Omega^1(N) \longrightarrow \Omega^1(M)$$

This can be extended to tensor spaces, and to k-forms in particular, by taking

$$\phi^* : \Omega^l(N) \longrightarrow \Omega^l(M)$$

where

$$\langle\, \phi^*(\omega);\ X_1, ..., X_l \,\rangle\,(a) \stackrel{def}{=} \Big\langle \omega_{\phi(a)};\ \phi_{*,a}(X_{1,a}), ..., \phi_{*,a}(X_{l,a}) \Big\rangle$$

for any $\omega \in \Omega^l(N)$, $X_i \in \mathcal{D}(M)$.
This homomorphism commutes with the differential d, and

$$\phi^*(\theta \wedge \omega) = \phi^*(\theta) \wedge \phi^*(\omega)$$

for any $\theta \in \Omega^s(N)$, $\omega \in \Omega^t(N)$.

If we have two manifolds M and N equipped with the involutive distributions P and Q respectively, what should it mean that (M, P) maps to (N, Q)? We make the following step.

Definition 1.10 *Given pairs (M, P) and (N, Q), where M and N are manifolds, P and Q involutive distributions on M and N respectively, then we say that a smooth map $\phi : M \longrightarrow N$ is a* **morphism of distributions** *if*

$$\phi^*(\, Ann(Q)\,) \subset Ann(P) \tag{1.1}$$

where ϕ^* is the induced homomorphism $\phi^* : \Omega^1(N) \longrightarrow \Omega^1(M)$.

We will sometimes write $(M, P) \xrightarrow{\phi} (N, Q)$.
Geometrically, this means that a vector tangent to P is mapped into a vector tangent to Q by the differential of ϕ :
For any $a \in M$, $\phi_{*,a}(P_a) \subset Q_b$ where $b = \phi(a)$.

Proposition 1.3 *The restriction of ϕ^* on $\Omega^l(Q) \subset \Omega^l(N)$ gives a $\mathcal{F}(P)$-module homomorphism*

$$\phi^* : \Omega^l(Q) \longrightarrow \Omega^l(P)$$

Proof: Let ω be an element of $\Omega^l(Q)$, that is,

$$\Delta(Q) \lrcorner\ \omega = 0 \quad and \quad \Delta(Q) \lrcorner\ d\omega = 0$$

Take any $X \in \Delta(P)$. Then

$$X_a \lrcorner\ \phi_a^*(\omega_{\phi(a)}) = \phi_{*,a}(X_a) \lrcorner\ \omega_{\phi(a)} = 0$$

for any $a \in M$ since $\phi_{*,a}(X_a) \in Q_{\phi(a)}$. Likewise,

$$X_a \lrcorner\ \phi_a^*(d\omega_{\phi(a)}) = \phi_{*,a}(X_a) \lrcorner\ d\omega_{\phi(a)} = 0$$

for any $a \in M$ for the same reason. Thus, $\phi^*(\omega) \in \Omega^l(P)$. □

Corollary 1.2 *A morphism of distributions $\phi : (M, P) \longrightarrow (N, Q)$ induces a graded algebra-homomorphism*

$$\phi^* : \Omega^\cdot(Q) \longrightarrow \Omega^\cdot(P)$$

by restriction to $\Omega^\cdot(Q) \subset \Omega^\cdot(N)$ which commutes with $\tilde{d}$

$$\phi^* \circ \tilde{d} = \tilde{d} \circ \phi^*$$

Proof : All homomorphism-properties of ϕ^* are clear, and the **Proposition 1.3** demonstrates that ϕ^* maps $\Omega^l(Q)$ into $\Omega^l(P)$ for each l. □

1.7 Cohomologies of solution spaces.

We now have the $\mathbb{Z}$-graded algebra $\Omega^{\cdot}(P)$ connected to our distribution, together with the derivation $\tilde{d} : \Omega^{\cdot}(P) \to \Omega^{\cdot}(P)$. Thus we have the following sequence:

$$0 \longrightarrow \mathcal{F}(P) \xrightarrow{\tilde{d}} \Omega^1(P) \xrightarrow{\tilde{d}} \ldots\ldots \xrightarrow{\tilde{d}} \Omega^k(P) \longrightarrow 0 \tag{1.2}$$

We will denote $\tilde{d} = \tilde{d}_l : \Omega^l(P) \longrightarrow \Omega^{l+1}(P)$.

Definition 1.11

$$Z^l(P) = Ker(\tilde{d}_l)$$

$$B^l(P) = Im(\tilde{d}_{l-1})$$

We know that $\tilde{d}^2 = 0$, thus $B^l(P)$ is a linear subspace of $Z^l(P)$ for any l. We form the following quotient

Definition 1.12 *For every $l \in \mathbb{Z}$ we define*

$$H^l(P) = Z^l(P)/B^l(P)$$

to be the **l-th cohomology group of the solution space of P**.

It is clear that $H^l(P) = \{0\}$ for $l < 0$ and $l > k$.
We can take the direct sum

$$H^{\cdot}(P) = \bigoplus_{l\in\mathbb{Z}} H^l(P) = Z^{\cdot}(P)/B^{\cdot}(P)$$

where $Z^{\cdot}(P) = \bigoplus_{l\in\mathbb{Z}} Z^l(P)$ and $B^{\cdot}(P) = \bigoplus_{l\in\mathbb{Z}} B^l(P)$.

Theorem 1.4 *$H^{\cdot}(P)$ is a $\mathbb{Z}$-graded super-symmetric algebra with respect to the multiplication induced by the wedge product:*

$$[\,\theta\,] \wedge [\,\omega\,] \overset{def}{=} [\,\theta \wedge \omega\,]$$

for $\theta \in Z^r(P)$, $\omega \in Z^s(P)$.

Proof: We have to show that the product is well-defined. Thus we prove that
(i) $Z^r(P) \wedge Z^s(P) \subset Z^{r+s}(P)$ and
(ii) $Z^r(P) \wedge B^s(P) \subset B^{r+s}(P)$ for all r, s.

Assume that $\theta \in Z^r(P)$ and $\omega \in Z^s(P)$.
Then $\tilde{d}(\theta \wedge \omega) = \theta \wedge d\omega + (-1)^r \theta \wedge d\omega = 0$, which proves (i).

Let $\theta \in Z^r(P)$ and $\omega \in B^s(P)$.
That is, $d\theta = 0$, and there exists a $\omega_0 \in \Omega^{s-1}(P)$ such that $\omega = d\omega_0$. But then we have that $d(\omega_0 \wedge \theta) = d\omega_0 \wedge \theta + (-1)^{s-1}\omega_0 \wedge d\theta = \omega \wedge \theta$, so $\omega \wedge \theta$ is in the image of $\tilde{d}_{s+r-1}$, which concludes the proof of (ii).
The rest of the properties are inherited from the wedge product. □

We shall investigate a slightly different picture. Namely, what happens if we have a morphism of distributions, say

$$\phi : (M, P) \longrightarrow (N, Q)$$

Can we connect their cohomology groups? The answer to that is yes. We know that ϕ generates a homomorphism from $\Omega^l(Q)$ to $\Omega^l(P)$. Thus we may investigate what happens to the subspaces $Z^l(Q)$ and $B^l(Q)$, and in turn if ϕ^* can be extended to a map of cohomology groups.

Lemma 1.1 *For any $l \in \mathbb{Z}$ we have that*

$$\phi^*(Z^l(Q)) \subset Z^l(P) \quad \text{and} \quad \phi^*(B^l(Q)) \subset B^l(P)$$

Proof : Let $\omega \in Z^l(Q) \subset \Omega^l(Q)$. Then $\phi^*(\omega) \in \Omega^l(P)$, and

$$\tilde{d}(\phi^*(\omega)) = d(\phi^*(\omega)) = \phi^*(d\omega) = 0$$

which proves the first part of the lemma. Now assume that $\theta \in B^l(Q)$. Then there exists a $\theta_0 \in \Omega^{l-1}(Q)$ so that $d\theta_0 = \theta$, and

$$\phi^*(\theta) = \phi^*(d\theta_0) = d(\phi^*(\theta_0))$$

This implies that $\phi^*(\theta)$ is the image of the element $\phi^*(\theta_0) \in \Omega^{l-1}(P)$, which completes the proof. □

Corollary 1.3 *A morphism of distributions* $\phi : (M, P) \longrightarrow (N, Q)$ *induces a graded algebra-homomorphism*

$$\phi^* : H^{\cdot}(Q) \longrightarrow H^{\cdot}(P)$$

by $\phi^*([\,\theta\,]) \stackrel{\text{def}}{=} [\,\phi^*(\theta)\,]$

1.8 Examples

The following example will illustrate a distribution with a fairly complicated solutionspace.

Example 1.1 Let $M = T^2$, the torus with angular coordinates (θ, ρ). It can be interpreted as the quotient $\mathbb{R}^2/\sim$ where $(\theta, \rho) \sim (\theta + 2\pi n, \rho + 2\pi m)$ for all $m, n \in \mathbb{Z}$. We will describe our torus as the quotient $D/\sim$ of the fundamental domain $D = \{(\theta, \rho) \in \mathbb{R}^2 \mid 0 \leq \theta, \rho \leq 2\pi\}$. Given $\omega \in \mathbb{R}\backslash\mathbb{Q}$, **the irrational flow** on the torus is the flow generated by the vectorfield

$$X_\omega = \partial_\theta + \omega\partial_\rho$$

This gives us a distribution P of dimension 1 and codimension 1 on T^2. When ω is an irrational number, each integral curve of the distribution is dense in T^2. Thus, this provides us with a rather complicated solution space $\mathcal{S}$ of the distribution. But it is possible to investigate $\mathcal{F}(P)$ and $\Omega^1(P)$ for our P. We have the following

$$\mathcal{F}(P) = \left\{ f \in C^\infty(\mathbb{R}^2) \;\middle|\; \begin{array}{l} f(\theta, \rho) = f(\theta + 2\pi n, \rho + 2\pi m) \quad \forall m, n \in \mathbb{Z} \\ and \quad f_\theta = -\omega f_\rho \end{array} \right\} \cong \mathbb{R}$$

since the only smooth functions that are constant on a dense set in $\mathbb{R}^2$ are the constant functions themselves.

$$\Omega^1(P) = \{\, f(-\omega d\theta + d\rho) \mid f \in \mathcal{F}(P)\} = \mathbb{R}\,\{\, -\omega d\theta + d\rho \,\}$$

and the sequence

$$0 \longrightarrow \mathcal{F}(P) \xrightarrow{d_0} \Omega^1(P) \longrightarrow 0$$

with $H^0(P) = ker\, d_0 = \mathcal{F}(P) = \mathbb{R}$

In order to illustrate the complicated geometry of $\mathcal{S}$, we will make a few pictures. Starting with the fundamental domain D, we see that we can describe $\mathcal{S}$ in terms of a smaller fundamental domain for the equivalence relation $\sim$ on D.

If $\omega < 0$, then all equivalence classes of $\sim$ are represented on the line $(\theta, -\theta)$, $\theta \in [0, 2\pi]$ in D, and this line can again be identified with the interval $I = [0, 2\pi]$. If $\omega > 0$, do the same for the line (θ, θ). Each integral curve of X_ω leaves a dense set of crossing points on this line (see figure below) that should all be identified by $\sim$. Thus, we get our space $\mathcal{S}$ by identifying all points on I that correspond to *one* integral curve in D. This will be a very complicated picture since each curve contributes with a dense set of points in I.

Figure 1.2
Illustration of $\mathcal{S} = I/\sim$ as quotient space of smallest fundamental domain.

We include yet another example of a 1-dimensional distribution on a two dimensional manifold.

Example 1.2 Let $M = \mathbb{R}^2 \backslash \{0\}$ with coordinates x, y, and P be the distribution generated by the single vectorfield $X = \partial_y$. The integral curve through any point of M is obviously the vertical line through the point. Each point on the x-axis with $x \neq 0$ represents a point in S. But the points $(0, y)$ divide into two classes, one class for $y > 0$, and one for $y < 0$. Thus, our solution space is the so-called "line with double point". It looks like a copy of $\mathbb{R}\backslash\{0\}$ with two points added at the origin, upper point denoted a, lower point denoted b.

Figure 1.3 *Line with double point.*

It is a non-Hausdorff manifold, covered by the following smooth charts. Take

$$\mathcal{U}_a = S \setminus \{b\}$$

and chart map ϕ_a by

$$Id_{\mathbb{R}\setminus\{0\}} : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R} \quad \text{and} \quad a \mapsto 0 \in \mathbb{R}$$

Similarly we define a chart $(\mathcal{U}_b\,,\,\phi_b)$ by changing the roles of the points. These charts cover S, and on their intersection $\mathcal{U}_a \cap \mathcal{U}_b = \mathbb{R}\setminus\{0\}$ the chart-maps are equal, and thus compatible. We see that it is thus impossible to differ between upper and lower point with continous functions. In these charts, the smooth functions are smooth functions on the line that have the same value at a an b. Both $\mathcal{U}_a$ and $\mathcal{U}_b$ are isomorphic to $\mathbb{R}$.

We want to calculate $\mathcal{F}(P)$, $\mathcal{D}(P)$ and $\Omega^1(P)$.

$$\mathcal{F}(P) = \{\, f \in C^\infty(M) \mid X(f) = \partial_y(f) = 0 \,\} \cong C^\infty(\mathbb{R})$$
$$\Omega^1(P) \cong \Omega^1(\mathbb{R})$$

since $\omega = f(x,y)\,dx + g(x,y)\,dy \in \Omega^1(P)$ has to satisfy

$$(i) \quad \langle \omega, \partial_y \rangle = 0 \quad \text{and} \quad (ii) \quad d\omega \lrcorner\, \partial_y = \partial_y(f) dy \wedge dx \lrcorner\, \partial_y = 0$$

which implies that $f = f(x) \in C^\infty(\mathbb{R})$ and $g \equiv 0$, hence $\omega = f(x)dx$.

$$\mathcal{D}(P) \cong \mathcal{D}(\mathbb{R})$$

since any element $Y = f(x,y)\partial_x + g(x,y)\partial_y$ in $Sym(P)$ has to satisfy

$$[\,\partial_y,\, f(x,y)\partial_x + g(x,y)\partial_y\,] = h\,\partial_y$$

for some function $h = h(x,y)$. This gives that $f = f(x)$, smooth function. Then $Y \mod X = f(x)\partial_x$ is the form of any element of $\mathcal{D}(P)$.

Chapter 2

Riemannian structures on solution spaces.

2.1 Symmetric 2-forms on $\mathcal{S}$

We wish to investigate if it is possible to make further constructions similar to usual tensor calculus over our space $\mathcal{S}$. In this section we will look for the analogue of the Riemannian manifold, namely a space equipped with a symmetric (positive definite) 2-form.

Definition 2.1 *We define*

$$S^2(P) = \left\{ \theta \in S^2(\Omega^1(M)) \;\middle|\; \begin{array}{r} \theta(X,\,\cdot\,) = 0 \quad and \\ L_X(\theta) = 0 \quad \forall X \in \Delta(P) \end{array} \right\}$$

We shall see that an element of $S^2(P)$ provides us with the following map:

Definition 2.2 *Given $g \in S^2(P)$ we define*

$$\tilde{g} : \mathcal{D}(P) \times \mathcal{D}(P) \longrightarrow \mathcal{F}(P)$$

by $\tilde{g}(\overline{X}_1, \overline{X}_2) = g(X_1, X_2)$, where X_1, X_2 are any representatives of $\overline{X}_1, \overline{X}_2 \in \mathcal{D}(P)$.

It follows from the definition of $S^2(P)$ that the mapping is well-defined: $g(X + X_0,\,\cdot\,) = g(X,\,\cdot\,)$ for any $X_0 \in \Delta(P)$. What we need to check, is that $g(X_1, X_2) \in \mathcal{F}(P)$ for any $X_1, X_2 \in Sym(P)$. For every $X \in \Delta(P)$ we have

$$L_X(g(X_1, X_2)) = L_X(g)(X_1, X_2) + \gamma_g$$

where $\gamma_g = g(L_X X_1, X_2) + g(X_1, L_X X_2) = 0$, since $L_X X_1, L_X X_2 \in \Delta(P)$. In addition , $L_X(g) = 0$, so $L_X(g(X_1, X_2)) = 0$ which implies $g(X_1, X_2) \in \mathcal{F}(P)$.

Remark : The mapping is symmetric, bilinear and $\mathcal{F}(P)$-additive.

Proposition 2.1 *Any element $g \in S^2(P)$ is on the form*

$$g = \sum_{1 \leq i \leq j \leq k} a_{ij} \theta_i \cdot \theta_j$$

where $\{\theta_i\}_{i=1}^k$ are generators of $Ann(P)$, and a_{ij}-s some functions.

The requirement $L_\Delta(g) = 0$ will give requirements on the a_{ij}-s. A specific calculation of this will be done in the **Section 3.2**, for the Cartan distribution.

2.2 Connection and curvature of $(\mathcal{S}, \tilde{g})$

Assume we have a symmetric 2-form $g \in S^2(P)$. We say that g is **positive** if it satisfies the condition

$$g(X_a, X_a) > 0 \quad \text{for any } X_a \notin P_a\ ,\ a \in M$$

For an ordinary Riemannian manifold (M, g), we have the unique Levi-Civita (Riemannian) connection (see **Example ??** in Section **??**). We wish to do a similar construction for our space $(\mathcal{S}, \tilde{g})$.

Definition 2.3 *We say that a map*

$$\nabla : \mathcal{D}(P) \times \mathcal{D}(P) \longrightarrow \mathcal{D}(P)$$

with $\nabla(\overline{X}, \overline{Y}) = \nabla_{\overline{X}}(\overline{Y})$ is a **connection on the solution space of P** *if it satisfies the following:*

(1) $\nabla_{\overline{X}}(f\overline{Y} + g\overline{Z}) = f\nabla_{\overline{X}}\overline{Y} + g\nabla_{\overline{X}}\overline{Z} + \overline{X}(f) \cdot \overline{Y} + \overline{X}(g) \cdot \overline{Z}$

(2) $\nabla_{f\overline{X} + g\overline{Y}}\overline{Z} = f\nabla_{\overline{X}}\overline{Z} + g\nabla_{\overline{Y}}\overline{Z}$

for any $\overline{X}, \overline{Y}$ and $\overline{Z} \in \mathcal{D}(P)$, $f, g \in \mathcal{F}(P)$.
If our space $\mathcal{S}$ is equipped with a positive $g \in S^2(P)$ we say that ∇ is a **Levi -Civita connection** *if it in addition satisfies the properties* (3) *and* (4) *bleow*

(3) $\left[\overline{X}, \overline{Y}\right] = \nabla_{\overline{X}}\overline{Y} - \nabla_{\overline{Y}}\overline{X}$

(4) $\overline{X}\left(\tilde{g}(\overline{Y}, \overline{Z})\right) = \tilde{g}(\nabla_{\overline{X}}\overline{Y}, \overline{Z}) + \tilde{g}(\overline{Y}, \nabla_{\overline{X}}\overline{Z})$

To produce such a ∇, we assume that we have a connection on M, that is, an operator

$$\widetilde{\nabla} : \mathcal{D}(M) \times \mathcal{D}(M) \longrightarrow \mathcal{D}(M)$$

that satisfies properties $(1) - (2)$. To define ∇ we take

$$\nabla_{\overline{X}}(\overline{Y}) \stackrel{def}{=} \overline{\widetilde{\nabla}_X(Y)}$$

where X, Y are representatives of $\overline{X}$ and $\overline{Y}$. We obviously need this to be invariant with respect to choice of representatives X, Y.
Our first step will be to define our connection on a local basis of $\mathcal{D}(P)$, and then expand by properties (1) and (2).

STEP 1 We need local basises for both differential forms and vectorfields. To get an appropriate basis we do the following. Locally we can always find k 1st-integrals of our distribution, $f_1, ..., f_k$. We choose coordinates $x_1 = f_1, ..., x_k = f_k, x_{k+1}, ..., x_{k+n}$ so that $\{dx_i\}_{i=1}^{k+n}$ constitutes our local basis of $\Omega^1(M)$, and $\{\partial_i\}_{i=1}^{k+n}$ the dual basis for $\mathcal{D}(M)$. This assures that $\{\partial_i\}_{i=k+1}^{k+n}$ is a local basis of the module $\Delta(P)$, and that $\{dx_i\}_{i=1}^{k}$ is a local basis of the annihilating module $Ann(P)$. Also, a local basis for $\mathcal{D}(P)$ is given by the classes of $\{\partial_i\}_{i=1}^{k}$. We define the functions Γ_{ij}^s, $1 \leq s \leq k$ and $1 \leq i, j \leq k+n$ by the following equation

$$\widetilde{\nabla}_{\partial_i}(\partial_j) = \sum_{s=1}^{k} \Gamma_{ij}^s \partial_s + X_{ij} \tag{2.1}$$

where $X_{ij} \in \Delta(P)$. Then

$$\nabla_{\overline{\partial}_i}(\overline{\partial}_j) = \overline{\widetilde{\nabla}_{\partial_i}(\partial_j)} = \sum_{s=1}^{k} \Gamma_{ij}^s \overline{\partial}_s$$

For our construction to work we need our Γ_{ij}^s-s to satisfy several conditions. First of all, they should be elements of $\mathcal{F}(P)$. We shall see that this will not present a problem to us when all other conditions are satisfied. Second, we need that ∇ is well-defined. This is ensured if and only if we take

$$\Gamma_{ij}^s = 0 \quad \text{for} \;\; k < i \leq k+n \;\; \text{or} \;\; k < j \leq k+n$$

This immediatly gives that

$$\nabla_{\overline{X}}(0) = \nabla_0(\overline{Y}) \;\; \text{for all} \;\; \overline{X}, \overline{Y} \in \mathcal{D}(P)$$

STEP 2 To require that ∇ satisfies properties (3) and (4) will give us the final conditions on the Γ_{ij}^s-s.

We should have

$$(3) \quad [\overline{\partial}_i, \overline{\partial}_j] = \nabla_{\overline{\partial}_i}(\overline{\partial}_j) - \nabla_{\overline{\partial}_j}(\overline{\partial}_i)$$

for $1 \leq i, j \leq k$. But our local basis $\{\partial_i\}$ commutes, thus

$$[\overline{\partial}_i, \overline{\partial}_j] = 0 = \sum_{s=1}^{k} (\Gamma_{ij}^s - \Gamma_{ji}^s)\overline{\partial}_s$$

which is equivalent to the condition

$$(3) \quad \Gamma_{ij}^s = \Gamma_{ji}^s$$

In order to simplify some of our further calculations we will introduce the functions $g_{ij} = \tilde{g}(\overline{\partial}_i, \overline{\partial}_j) = g(\partial_i, \partial_j) \in \mathcal{F}(P)$. Our g is positive, so the matrix $G = (g_{ij})$ is invertible. The functions g^{ij} are the entrys of the inverse matrix, $G^{-1} = (g^{ij})$. Defining $\Gamma_{ijl} = \sum_{s=1}^{k} \Gamma_{ij}^s g_{sl}$ gives us the following reformulation of (3)

$$(3)' \quad \Gamma_{ijl} = \Gamma_{jil}$$

The condition (4) becomes

$$(4)' \quad \partial_i(g_{jl}) = \Gamma_{ijl} + \Gamma_{ilj}$$

Combining the two equations for different permutations of indices makes us arrive at the following

Lemma 2.1

$$\Gamma_{ijl} = \frac{1}{2}\left[\partial_j(g_{il}) - \partial_l(g_{ij}) + \partial_i(g_{jl})\right]$$

This means that our functions Γ_{ijl} are completely determined by g in a given *local* basis $\{\overline{\partial}_i\}$ of $\mathcal{D}(P)$. This in turn determines the functions

$$\Gamma^s_{ij} = \sum_{l=1}^{k} \Gamma_{ijl} g^{ls}$$

and thus our connection ∇ locally. In ordinary Riemannian geometry the functions Γ^s_{ij} are called the Christoffel symbols, and an approach to find that they are uniquely determined by g is found in [Boo75]. That approach has been the model of the construction above.

Remark : It is clear that every Γ^s_{ij} is an element of $\mathcal{F}(P)$. All g_{ij} and g^{ij}-s are elements of $\mathcal{F}(P)$, and all elements $\overline{\partial}_i \in \mathcal{D}(P)$ used on elements of $\mathcal{F}(P)$ will give elements of $\mathcal{F}(P)$. Thus, the Γ^s_{ij}, which are sums and products of such functions, will be elements of $\mathcal{F}(P)$. This concludes our construction.

Theorem 2.1 *Let P be an involutive distribution, and $\mathcal{S}$ its solution space. For any positive $g \in S^2(P)$ there is a uniquely determined Levi-Civita connection ∇ on the solution space of P.*

Proof : The **Lemma 2.1** proves that ∇ is uniquely determined locally in terms of the functions Γ_{ijl}, for any such g.

This procedure has shown the existence of a connection ∇ on $\mathcal{S}$. It can also be copied in a basis different from the one above.

Assume that we have a k-dimensional transversal Lie-algebra $\mathcal{G} \subset \mathcal{D}(P)$ with generators $\{\overline{X}_i\}_{i=1}^k$. Transversality is equivalent to the condition that the matrix $(\theta_i(X_j))$ is nondegenerate, where $\{\theta_i\}_{i=1}^k$ are generators of $Ann(P)$. The generators have the following commutator:

$$\left[\overline{X}_i, \overline{X}_j\right] = \sum_{s=1}^{k} c^s_{ij} \overline{X}_s$$

We can find a basis $\{X_i\}_{i=1}^{k+n}$ of $\mathcal{D}(M)$ so that $\{X_i\}_{i=1}^k$ is a set of representatives of $\{\overline{X}_i\}$. As in the approach above we define the functions Γ^s_{ij},

$1 \leq i, j \leq n+k$, by the equations

$$\widetilde{\nabla}_{X_i} X_j = \sum_{s=1}^{k} \Gamma_{ij}^{s} X_s + Y_{ij}$$

where $Y_{ij} \in \Delta(P)$. As before, we have

$$\nabla_{\overline{X}_i} \overline{X}_j = \overline{\widetilde{\nabla}_{X_i} X_j} = \sum_{s=1}^{k} \Gamma_{ij}^{s} \overline{X}_s$$

and we must choose $\Gamma_{ij}^{s} = 0$ for $k < i \leq k+n$ or $k < j \leq k+n$ to ensure that ∇ is well-defined.

To find the remaining Γ_{ij}^{s}-s we again investigate the properties (3) and (4) and get the following conditions. (3) is equivalent to

$$(3)' \quad \Gamma_{ijl} = \Gamma_{jil} + c_{ijl}$$

where $c_{ijl} = \sum_{s=1}^{k} c_{ij}^{s} g_{sl}$ The condition (4) becomes

$$(4)' \qquad X_i(g_{jl}) = \Gamma_{ijl} + \Gamma_{ilj}$$

Again, combining the two equations for different permutations of indices makes us arrive at the following:

Theorem 2.2

$$\Gamma_{ijl} = \frac{1}{2}\left[c_{ijl} - c_{lji} + c_{lij} + X_j(g_{il}) - X_l(g_{ij}) + X_i(g_{jl})\right]$$

for $1 \leq i, j, l \leq k$.

This gives us our connection in terms of the basis $\{\overline{X}_i\}$ and g, in the interesting case when we have a Lie-algebra of symmetries of our distribution.

We are now ready to define curvature of our ∇.

Definition 2.4 *The* **curvature** R_∇ *of* ∇ *is defined by*

$$R_\nabla(\overline{X}, \overline{Y}) = [\nabla_{\overline{X}}, \nabla_{\overline{Y}}] - \nabla_{[\overline{X}, \overline{Y}]}$$

R_∇ has the following properties.

Proposition 2.2 *For any $\overline{X}, \overline{Y} \in \mathcal{D}(P)$ we have*

(1) $R_\nabla(\overline{X}, \overline{Y}) = -R_\nabla(\overline{Y}, \overline{X})$

(2) R_∇ *is* $\mathcal{F}(P)$*-linear in both arguments.*

(3) $R_\nabla(\overline{X}, \overline{Y})$ *is a* $\mathcal{F}(P)$*-module homomorphism of* $\mathcal{D}(P)$*.*

Proof : Property (1) is obvious, (2) and (3) will follow by direct calculation.

Thus, we can think of R_∇ as an element of $\Omega^2(P) \otimes End(\mathcal{D}(P))$.

Definition 2.5 *We define the operator*

$$R : \mathcal{D}(P) \times \mathcal{D}(P) \times \mathcal{D}(P) \times \mathcal{D}(P) \longrightarrow \mathcal{F}(P)$$

by

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \tilde{g}\left(R_\nabla(\overline{X}, \overline{Y})(\overline{Z}), \overline{W}\right)$$

The operator R satisfies the following properties.

Proposition 2.3 *For any $\overline{X}, \overline{Y}, \overline{Z}, \overline{W} \in \mathcal{D}(P)$ we have*

(1) $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -R(\overline{Y}, \overline{X}, \overline{Z}, \overline{W})$

(2) R *is* $\mathcal{F}(P)$*-linear in all arguments.*

Proof : The properties follow directly from the properties of R_∇ and $\tilde{g}$.

Chapter 3

ODE-s as distributions

3.1 The Cartan distribution $\mathcal{C}_\mathcal{E}$

We wish to investigate ordinary differential equations of order k, preferably those that can be resolved with respect to the highest derivative:

$$y^{(k)} = F(x, y, y', ..., y^{(k-1)}) \tag{3.1}$$

We connect this equation to a special distribution in the space $J^k\mathbb{R}$ in the following way.
$J^k\mathbb{R}$ is a $(k+2)$-dimensional space whose points are all $(k+2)$-tuples $(a, y(a), y'(a), ..., y^{(k)}(a))$ with $a \in \mathbb{R}$, $y \in C^\infty(\mathbb{R})$. Thus we can take $J^k\mathbb{R} = \mathbb{R}^{k+2}$ with coordinates $x, p_0, ..., p_k$. In $J^k\mathbb{R}$ we have the *Cartan distribution* $\mathcal{C}$ of dimension 2 and codimension k defined by the k 1-forms

$$\begin{aligned} \omega_0 &= dp_0 - p_1 dx \\ \omega_1 &= dp_1 - p_2 dx \\ &\vdots \\ \omega_{k-1} &= dp_{k-1} - p_k dx \end{aligned} \tag{3.2}$$

Moreover, our equation 3.1 defines a hypersurface

$$\mathcal{E} = \{ \ (x, p_0, p_1, ..., p_k) \in J^k\mathbb{R} \mid p_k - F(x, p_0, p_1, ..., p_{k-1}) = 0 \ \}$$

in $J^k\mathbb{R}$ on which we take coordinates $x, p_0, ..., p_{k-1}$. The distribution $\mathcal{C}$ produces a 1-dimensional distribution $\mathcal{C}_\mathcal{E}$ when restricted to $\mathcal{E}$, and we call this the *Cartan distribution of the equation* $\mathcal{E}$. It has the same defining forms, except we take

$$\omega_{k-1} = dp_{k-1} - F(x, p_0, ..p_{k-1})dx$$

Integral curves of $\mathcal{C}_{\mathcal{E}}$ that project to the x-axis without degeneration can only be curves of type

$$p_0 = y(x),\ p_1 = y'(x),\ \dots, p_k = y^{(k)}(x)$$

in $J^k\mathbb{R}$ where $y(x) \in C^\infty(\mathbb{R})$ is a **solution** of 3.1. The latter requirement ensures that the curve lies in $\mathcal{E}$. Thus finding solutions to the original differential equation is equivalent to finding integral curves of the distribution $\mathcal{C}_{\mathcal{E}}$, and this will be our opening towards the constructions presented in **Section 1.2**.

3.2 Symmetries and cosymmetries of the Cartan distribution

We start with the investigation of $\mathcal{D}(\mathcal{C}_{\mathcal{E}})$, and include ideas and results from [DL91].
First of all, we note that the $C^\infty(\mathbb{R}^{k+1})$-module $\Delta(\mathcal{C}_{\mathcal{E}})$ is generated by a single vectorfield

$$D = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial p_0} + \dots + p_{k-1}\frac{\partial}{\partial p_{k-2}} + F\frac{\partial}{\partial p_{k-1}} \tag{3.3}$$

In the quotient algebra $Shuf(\mathcal{C}_{\mathcal{E}})$ we have the relation

$$\frac{\partial}{\partial x} \equiv -p_1\frac{\partial}{\partial p_0} - \dots - p_{k-1}\frac{\partial}{\partial p_{k-2}} - F\frac{\partial}{\partial p_{k-1}} \quad \bmod Char(\mathcal{C}_{\mathcal{E}})$$

So we are only looking for symmetries on the form

$$X = \alpha_0\frac{\partial}{\partial p_0} + \dots + \alpha_{k-2}\frac{\partial}{\partial p_{k-2}} + \alpha_{k-1}\frac{\partial}{\partial p_{k-1}}$$

Combining this, and employing **Theorem 1.2** by investigating the Lie-derivatives $L_D(\omega_i),\ \ i = 0..k-1$, we arrive at the following. Any symmetry (modulo $\mathrm{Char}(\mathcal{C}_{\mathcal{E}})$) has to be on the form

$$X = X_\phi = \sum_{i=0}^{k-1} D^i(\phi)\frac{\partial}{\partial p_i} \tag{3.4}$$

where the function $\phi = \phi(x, p_0, ..., p_{k-1})$ solves the **Lie equation**

$$L(\phi) = 0 \tag{3.5}$$

for

$$L = D^k - \sum_{i=0}^{k-1} \frac{\partial F}{\partial p_i} D^i \tag{3.6}$$

the **Lie operator**. The function ϕ is called the **generating function** of the symmetry X_ϕ. The results above are shown more rigorously in the article [DL91]. The article mainly treats the problem of finding symmetries of the distribution by the approach explained above, and thereby finding solutions (1st-integrals) by quadrature. We will try to expand this picture by including some of our constructions.

As our distribution $\mathcal{C}_\mathcal{E}$ is 1-dimensional, it is completely integrable by Theorem 1.1 (the Frobenius theorem). We wish to place $\mathcal{C}_\mathcal{E}$ in the setting of Section 1.2. According to our definitions 1st-integrals are

$$\mathcal{F}(\mathcal{C}_\mathcal{E}) = \left\{ f \in C^\infty(\mathbb{R}^{k+1}) \mid L_D(f) = 0 \right\}$$

Furthermore, differential 1-forms are

$$\Omega^1(\mathcal{C}_\mathcal{E}) = \left\{ \theta \in \Omega^1(\mathcal{E}) \mid D \lrcorner \theta = 0 \;\; and \;\; D \lrcorner d\theta = 0 \right\}$$

As a matter of fact, the forms in $\Omega^1(\mathcal{C}_\mathcal{E})$ are also given in terms of a generating function , as is the case for symmetries modulo characteristic symmetries. To recognize this we have to make the following steps.

The requirement $D \lrcorner \theta = 0$ implies that θ has to be on the form

$$\theta = \sum_{i=0}^{k-1} \beta_i \omega_i$$

for some functions β_i since the Cartan forms $\{\omega_i\}$ generate the annihilating module $Ann(P)$. Combining that, and the requirement $L_D(\theta) = 0$ we get exactly the forms in $\Omega^1(\mathcal{C}_\mathcal{E})$. By calculation we have

$$\begin{aligned} L_D(\omega_i) &= \omega_{i+1} \qquad i = 0...k-2 \\ L_D(\omega_{k-1}) &= \sum_{i=0}^{k-1} \frac{\partial F}{\partial p_i} \omega_i \end{aligned}$$

The requirement $L_D(\theta) = 0$ then gives the following set of $k-1$ equations on the functions β_i :

$$D(\beta_0) = -F_0\psi \tag{3.7}$$
$$\beta_j = -D(\beta_{j+1}) - \psi F_{j+1}\ , \quad j = 0...k-2 \tag{3.8}$$

where $F_i = \frac{\partial F}{\partial p_i}$ and $\psi = \beta_{k-1}$. It is clear that by combining these equations we get one equation on ψ alone, and the β_j-s are given from ψ. That is

$$\beta_{k-l} = H_l(\psi) \qquad l = 2...k \tag{3.9}$$

where H_l is the operator

$$H_l = (-1)^{l-1}D^{l-1} - \sum_{s=0}^{l-2}(-1)^s D^s f_s^l \tag{3.10}$$

with $f_s^l = F_{k-l+1+s}$.

Proof of equations **(3.9)** and **(3.10)**. We will use induction on l.
For **l=2** we have:

$$\beta_{k-2} = (-D - F_{k-1})\psi$$

from (3.8). Our formula (3.10) gives

$$H_2 = -D - F_{k-1}$$

which implies that (3.10) is correct for l = 2.
We will assume that it is also correct for l, and calculate $\beta_{k-(l+1)} = \beta_{k-l-1}$.

$$\beta_{k-l-1} = -D(\beta_{k-l}) - F_{k-l}\psi$$

by equation (3.8). Then

$$\beta_{k-l-1} = (-D \cdot H_l - F_{k-l})\,\psi$$

By direct calculation (note that $f_{r-1}^l = f_r^{l+1}$) we get that

$$\beta_{k-l-1} = \left[(-1)^l D^l - \sum_{s=0}^{l-1}(-1)^s D^s f_s^{l+1}\right]\psi$$

which is exactly $H_{l+1}\psi$. □

We can thus take $l = k$ and combine equations

$$\beta_0 = H_k(\psi)$$

and

$$D(\beta_0) + F_0\psi = 0$$

and get the following equation for ψ

$$L^*(\psi) = 0 \tag{3.11}$$

where

$$L^* = (-1)^k(D^k) - \sum_{r=0}^{k-1}(-1)^r D^r F_r \tag{3.12}$$

We can calculate the adjoint operator $Adj(L)$ of L by the rules $Adj(AB) = Adj(B)Adj(A)$ where $Adj(f) = f$ for functions, and $Adj(D) = -D$ else. This shows that L^* is actually equal to $Adj(L)$.

L^* can be written in the following way, ordered by the degree of D :

$$L^* = (-1)^k D^k - \sum_{r=0}^{k-1}(-1)^r \sum_{s=0}^{r} \binom{r}{s} D^s(F_r)D^{r-s} \tag{3.13}$$

Theorem 3.1 *Any element in $\Omega^1(\mathcal{C}_\mathcal{E})$ is on the form*

$$\theta = \theta_\psi = \psi\omega_{k-1} + \sum_{l=2}^{k} H_l(\psi)\omega_{k-l}$$

for some generating function ψ that solves equation (3.11), the H_l-s as given in equation (3.10)

We shall call such 1-forms **cosymmetries** of our distribution, as they are "duals" to our symmetries. We know that any pairing

$$f = \langle\, X_\phi, \theta_\psi \,\rangle$$

of a symmetry and a cosymmetry will give a 1st-integral of the distribution $\mathcal{C}_\mathcal{E}$. If we have k independent 1st-integrals of $\mathcal{C}_\mathcal{E}$, we have a implicit solution of our equation. In general, if we have some number of symmetries and cosymmetries, different pairing-combinations can give us functions that are just constants or constant-multiples of eachother. So, having k independent symmetries, and k independent cosymmetries is no guarantee to get k independent 1st-integrals.

3.3 Remarks on $S^2(\mathcal{C}_\mathcal{E})$

How to find elements of $S^2(P)$? We can try to find such symmetric 2-forms directly from our definition. Assume $g \in S^2(\Omega^1(M))$. The requirement $g(\Delta(P), \cdot) = 0$ implies that

$$g = \sum_{0 \le i,j \le k-1} a_{ij} \omega_i \cdot \omega_j$$

where $\{\omega_i\}_{i=0}^{k-1}$ are the Cartan forms. The matrix $A = (a_{ij})$ is symmetric. In addition we must have $L_D(g) = 0$. This will give us a set of equations on the functions a_{ij}. These can be combined to give one equation on a generating $a_{i_0 j_0}$, and the other $a_{ij} - s$ will be given by the generator in a way similar to (co)symmetries. This will be done for the Cartan distribution $\mathcal{C}_\mathcal{E}$ in the case $k = 2$.

On the other hand, if we know k independent elements $\theta_1, .., \theta_k$ of $\Omega^1(P)$, we get a symmetric, positive 2-form

$$g = \sum_{i=1}^{k} \theta_i^2$$

immediately.

Chapter 4

Algebraic picture -relations with $\mathcal{D}$-modules

4.1 A-module E with a derivation

Let A be a commutative algebra with a derivation $\delta : A \longrightarrow A$, and E an A-module. Assume that we have an operator

$$\delta : E \longrightarrow E$$

on our module such that

$$\delta(ax) = \delta(a)x + a\delta(x) \tag{4.1}$$

for any $a \in A$, $x \in E$.
We denote this as a pair (E, δ), and call E a **$\mathcal{D}$-module**.
We define

$$E^{\#} = \{x \in E \mid \delta x = 0\} \quad \text{and} \quad A^{\#} = \{a \in A \mid \delta a = 0\}$$

Proposition 4.1 *$E^{\#}$ is an $A^{\#}$-module.*

Assume we have two such pairs (E_1, δ) and (E_2, δ). They give us new A-modules with operators in the following ways

(1) Take the tensorproduct $E_1 \otimes E_2$, which is again an A-module. The operator δ extends to the tensor product in the following way

$$\delta(x_1 \otimes x_2) \stackrel{\text{def}}{=} \delta(x_1) \otimes x_2 + x_1 \otimes \delta(x_2)$$

on decomposable elements, and extend by property (4.1). Thus you get a new pair

$$(E_1 \otimes E_2, \delta)$$

(2) Following (1), taking tensorproducts of E with itself gives us constructions like
$(S^l(E), \delta)$, where

$$\delta(x_1 \cdot \ldots \cdot x_l) = \delta(x_1) \cdot ... \cdot x_l + ... + x_1 \cdot ... \cdot \delta(x_l)$$

or $(\bigwedge^l(E), \delta)$, where

$$\delta(x_1 \wedge ... \wedge x_l) = \delta(x_1) \wedge ... \wedge x_l + ... + x_1 \wedge ... \wedge \delta(x_l)$$

(3) Also, we can extend δ to the A-module $Hom_A(E_1, E_2)$ by taking

$$(\delta F)(x) \stackrel{\text{def}}{=} F(\delta(x)) - \delta(F(x))$$

for any $F \in Hom_A(E_1, E_2)$, and thus get a new pair $(Hom_A(E_1, E_2), \delta)$

The constructions above will provide us with several A-modules with induced operators starting only with one pair (E, δ).

We can define the dual

$$E^* = Hom_A(E, A)$$

The induced operator δ is then given by the formula in (3).

Also, we can construct various tensorspaces of E and E^* like in (1) and (2), including

$$E \otimes E^* \cong Hom_A(E, E)$$

The reason this piece of algebra is included, is the following observation.
All our constructions $\mathcal{F}(\mathcal{C}_\mathcal{E})$, $\mathcal{D}(\mathcal{C}_\mathcal{E})$, $\Omega^l(\mathcal{C}_\mathcal{E})$, $S^2(\mathcal{C}_\mathcal{E})$ in the previous section fits exactly into this algebraic picture. Namely, we take

$$A = C^\infty(J^1\mathbb{R}) \quad \text{and} \quad E = \mathcal{D}(J^1\mathbb{R})/D$$

where D is the characteristic vectorfield of the distribution $\mathcal{C}_{\mathcal{E}}$. The operator in question is the Lie-derivative along D, $\delta = L_D$. So, starting with the pair

$$(E, L_D) \tag{4.2}$$

we get all our well known tensorspaces.

- The dual space $E^* = Ann(\mathcal{C}_{\mathcal{E}})$
  with the usual Lie-derivative L_D as induced operator.
- $A^\# = \mathcal{F}(\mathcal{C}_{\mathcal{E}})$
- $E^\# = \mathcal{D}(\mathcal{C}_{\mathcal{E}})$
- $(E^*)^\# = \Omega^1(\mathcal{C}_{\mathcal{E}})$
- $S^2(E^*) = \{ \sum \alpha_{ij}\theta_i \cdot \theta_j \mid \theta_i, \theta_j \in Ann(\mathcal{C}_{\mathcal{E}}), \alpha_{ij} \in A \}$
  with the usual Lie-derivative L_D as induced operator.
- $S^2(E^*)^\# = S^2(\mathcal{C}_{\mathcal{E}})$
- $\bigwedge^l(E^*) = \left\{ \sum \alpha_{i_1..i_l}\theta_{i_1} \wedge ... \wedge \theta_{i_l} \mid \theta_{i_j} \in Ann(\mathcal{C}_{\mathcal{E}}), \alpha_{i_1..i_l} \in A \right\}$
  with the usual Lie-derivative L_D as induced operator.
- $\left(\bigwedge^l(E^*)\right)^\# = \Omega^l(\mathcal{C}_{\mathcal{E}})$

Chapter 5

ODE-s of order two

5.1 Symmetry, cosymmmetry and g-equations

Let our equation be of order two, and resolved with respect to the 2nd derivative.

$$y'' = F(x, y, y') \tag{5.1}$$

In this section we will find the equations connected to its (co)symmetries and other constructions from **Section 1.2**. The distribution $\mathcal{C}_{\mathcal{E}}$ is of dimension 1 and codimension 2. We denote the coordinates on $\mathcal{E}$ by x, u and p, thus our vectorfield D becomes

$$D = \partial_x + p\partial_u + F\partial_p$$

where $F = F(x, u, p)$. The Cartan-forms are

$$\omega_0 = du - pdx \ , \quad \omega_1 = dp - Fdx$$

Our **Lie equation** is the following

$$L(\phi) = 0 \tag{5.2}$$

with the operator L

$$L = D^2 - F_p D - F_u \tag{5.3}$$

Here F_u and F_p are the partial derivatives w.r.t. the variables u and p. If ϕ solves equation (5.2), it generates the symmetry

$$X_\phi = \phi\,\partial_u + D(\phi)\,\partial_p$$

To find cosymmetries we should find solutions of the equation

$$L^*(\psi) = 0 \tag{5.4}$$

where

$$L^* = D^2 + D \cdot F_p - F_u = D^2 + F_p D + [D(F_p) - F_u] \tag{5.5}$$

which is equal to the adjoint operator $Adj(L)$ of L. A solution ψ of (5.4) generates a cosymmetry

$$\omega_\psi = -(D + F_p)(\psi)\,\omega_0 + \psi\,\omega_1$$

Before we investigate some examples of second order equations, we will find the conditions on a symmetric 2-form to be an element of $S^2(P)$. We recall that g has to be on the form

$$g = a_{00}\omega_0^2 + 2a_{01}\omega_0 \cdot \omega_1 + a_{11}\omega_1^2$$

to ensure that $g(D, \cdot) = 0$. In addition we need to have that $L_D(g) = 0$. We do the calculation

$$\begin{gathered} L_D(g) = \\ [\,D(a_{00}) + 2F_u a_{01}\,]\ \omega_0^2 + \\ +2\,[\,a_{00} + (D + F_p)a_{01} + F_u a_{11}\,]\,\omega_0 \cdot \omega_1 + [\,(D + 2F_p)a_{11} + 2a_{01}\,]\,\omega_1^2 \end{gathered}$$

and see that $L_D(g) = 0$ is equivalent to the set of equations $(i) - (iii)$ on the functions a_{ij}

$$\begin{array}{lcl} (i) & D(a_{00}) + 2F_u a_{01} & = 0 \\ (ii) & a_{00} + (D + F_p)a_{01} + a_{11}F_u & = 0 \\ (iii) & 2a_{01} + (D + 2F_p)a_{11} & = 0 \end{array}$$

These equations can be written in the following way

$$\begin{array}{lcl} (i) & D(a_{00}) & = -2F_u a_{01} \\ (ii) & a_{00} & = -(D + F_p)a_{01} - F_u a_{11} \\ (iii) & a_{01} & = -\dfrac{1}{2}(D + 2F_p)a_{11} \end{array}$$

Equation (iii) gives us a_{01} expressed in terms of a_{11}, and this we can substitute into equation (ii). Thereby we have both a_{01} and a_{00} expressed in terms of the function a_{11}. By substitution in equation (i) we arrive at the following conditions:
The function a_{11} must solve the equation

$$L_{11}(a_{11}) = 0 \tag{5.6}$$

where L_{11} is the following operator

$$L_{11} \;=\; D^3 \;+\; [\,3F_p\,]\,D^2 \;+\; [\,5D(F_p) + 2F_p^2 - 4F_u\,]\,D +$$

$$+[\,2D^2(F_p) + 4F_pD(F_p) - 2D(F_u) - F_pF_u\,]$$

Furthermore, the functions a_{00} and a_{01} are given by

$$a_{00} = L_{00}(a_{11}) \tag{5.7}$$

with

$$L_{00} = \frac{1}{2}\left(D^2 + 3F_pD + 2[D(F_p) + F_p^2 - F_u]\right)$$

and

$$a_{01} = L_{01}(a_{11}) \tag{5.8}$$

with

$$L_{01} = -\frac{1}{2}(D + 2F_p)$$

Our calculations on g can be summarised in the following way.

Theorem 5.1 *Any $g \in S^2(\mathcal{C}_\mathcal{E})$ is on the form*

$$g = L_{00}(\eta)\,\omega_0^2 + 2L_{01}(\eta)\,\omega_0\cdot\omega_1 + \eta\,\omega_1^2$$

where η solves the Equation (5.6).

Remark :
If we have two solutions ψ_1, ψ_2 of the L^*-equation, this produces two cosymmetries θ_1 and θ_2. Their symmetric products θ_1^2, $\theta_1\cdot\theta_2$ θ_2^2, as well as their

sums, must be elements of $S^2(\mathcal{C}_\mathcal{E})$.
Hence, the functions ψ_1^2, $\psi_1\psi_2$ and ψ_2^2 must solve the equation

$$L_{11}\eta = 0$$

We call the operator L_{11} **the symmetric power of L^*.**

We now know how all elements of $\mathcal{D}(\mathcal{C}_\mathcal{E})$, $\Omega^1(\mathcal{C}_\mathcal{E})$ and $S^2(\mathcal{C}_\mathcal{E})$ must look for $k = 2$; they are all given by generating functions that have to satisfy certain equations. These equations are in some sense more complicated than our original one. But we are not looking for general solutions of them: for example, to find a g for some equation $y" = F$ it suffices to find one solution of the L_{11}-equation. Therefore we can look for generating functions of special types, for example linear in p.
Also, we can try to investigate the problem, starting in the other end. Assume we have a specific 2-form $g = a_{00}\omega_0^2 + 2a_{01}\omega_0\omega_1 + a_{11}\omega_1^2$. We can then try to find for which F-s these a_{ij}-s satisfy the conditions of Theorem 5.1. That is, we get a class of equations that possess this kind of structure.

5.2 Equations with extra structure

Investigating an equation along the lines of our constructions connected to the space $\mathcal{S}$ will at this point consist in searching for elements of $\mathcal{D}(\mathcal{C}_\mathcal{E})$, $\Omega^1(\mathcal{C}_\mathcal{E})$ and $S^2(\mathcal{C}_\mathcal{E})$. This will, in one way or the other, mean investigation of the L, L^* and L_{11}- equations. One might also find other things than solutions during such an investigation. One thing that was discovered conserning a wide class of equations is the following.

Theorem 5.2 *Given an equation*

$$y'' = \gamma(x)y' + \delta(x, y)$$

we have the following relation between the associated L and L^*-equations:

$$L(\phi) = 0 \quad \Leftrightarrow \quad L^*(e^\alpha \phi) = 0$$

where $\alpha = \alpha(x)$ is any function such that $\alpha'(x) = -\gamma(x)$. The functions $\gamma(x)$ and $\delta(x, y)$ are arbitrary.

Proof : Let $F = \gamma(x)p + \delta(x,u)$. Then, for any ϕ we have

$$L(\phi) = D^2(\phi) - \gamma D(\phi) - F_u \phi$$

For any $\alpha = \alpha(x)$ we have by straightforward calculation that

$$L^*(e^\alpha \phi) = e^\alpha \left[D^2(\phi) + (2\alpha' + \gamma) D(\phi) + (\,\alpha'' + (\alpha')^2 + \alpha'\gamma + \gamma' - F_u\,)\phi \right]$$

Choosing α such that $\alpha' = -\gamma$ implies that

$$(2\alpha' + \gamma) \;=\; -\gamma \qquad and \qquad (\,\alpha'' + (\alpha')^2 + \alpha'\gamma + \gamma' - F_u\,) \;=\; -F_u$$

thus for this choice of α we have that

$$L^*(e^\alpha \phi) = e^\alpha L(\phi)$$

which completes the proof. $\square$

Given an equation of the type in **Theorem 5.2** , we observe that we get the following map:

$$\phi \in kerL \quad \mapsto \quad e^\alpha \phi \in kerL^*$$

which in turn gives the corresponding map

$$\lambda : \mathcal{D}(\mathcal{C}_\mathcal{E}) \longrightarrow \Omega^1(\mathcal{C}_\mathcal{E}) \tag{5.9}$$

defined by $X_\phi \mapsto \theta_{e^\alpha \phi}$. This map is obviously $\mathbb{R}$-linear.

We know that an element $\Lambda = \sum_i c_i \sigma_i \otimes \theta_i \in \Omega^1(M) \otimes \Omega^1(M)$ with $\sigma_i, \theta_i \in \Omega^1(M)$ and $c_i \in C^\infty(M)$ corresponds to a $C^\infty(M)$-linear mapping

$$\Lambda : \mathcal{D}(M) \longrightarrow \Omega^1(M)$$

given by $X \mapsto \sum_i c_i \sigma_i(X) \theta_i$.

We want to find such a Λ that maps X_ϕ into $\theta_{e^\alpha \phi}$ for every ϕ that solves $L(\phi) = 0$. Calculating directly $\Lambda(X_\phi)$ we get that

$$\Lambda = -2e^\alpha \omega_0 \wedge \omega_1 \tag{5.10}$$

which is an element of $\Omega^2(M)$.

To check if Λ is an element of $\Omega^2(\mathcal{C}_\mathcal{E})$ we calculate the Lie-derivative of Λ along D

$$L_D(\Lambda) = e^\alpha(-\gamma + \gamma)\omega_0 \wedge \omega_1 = 0$$

Thus we have the following result.

Theorem 5.3 *Equations of the type*

$$y'' = \gamma(x)y' + \delta(x, y)$$

where γ and δ are arbitrary, possess a symplectic structure on the space of solutions given by the non-degenerated 2-form $\Lambda = -2e^\alpha\omega_0 \wedge \omega_1 \in \Omega^2(\mathcal{C}_\mathcal{E})$.

Remark:
We make the following observation about 1st-integrals. Assume we have an equation of the type in Theorem 5.2, with $F_p = \gamma(x)$. Assume $\alpha = \alpha(x)$ so that $\alpha' = -\gamma$, and that we have a pair of symmetries X_{ϕ_1}, X_{ϕ_2} and thus also cosymmetries $\theta_{e^\alpha\phi_1}, \theta_{e^\alpha\phi_2}$. Their pairings will be the following:

$$\langle X_{\phi_1}, \theta_{e^\alpha\phi_1}\rangle = \langle X_{\phi_2}, \theta_{e^\alpha\phi_2}\rangle = 0$$

and

$$\langle X_{\phi_1}, \theta_{e^\alpha\phi_2}\rangle \;=\; -\langle X_{\phi_2}, \theta_{e^\alpha\phi_1}\rangle \;=\; \phi_1 D(\phi_2) - D(\phi_1)\phi_2$$

Thus, this will only give us one (possibly) non-zero 1st-integral up to multiplication by constant.

5.3 Equations with constant coefficient g

Assume that our equation

$$y'' = F(x, y, y')$$

posesses a metric structure where g is of the following form

$$g = a_{00}\omega_0^2 + 2a_{01}\omega_0 \cdot \omega_1 + a_{11}\omega_1^2$$

where all the a_{ij}-s are **constants**, ω_0, ω_1 the Cartan forms. We want to find all equations that can have such a g. To do this we have to examine the equations for the a_{ij}-s.

Take $c = a_{11} \in \mathbb{R}$. Then the equation (5.8) for a_{01} becomes

$$a_{01} = -F_p a_{11} = -F_p c$$

which implies that F_p is a constant

$$(i) \quad F_p = -\frac{a_{01}}{c}$$

If we look at the equation (5.7) for a_{00} we get that

$$a_{00} = (F_p^2 - F_u)c$$

which gives us the following requirement on F_u

$$(ii) \quad F_u = \frac{(a_{01}^2 - ca_{00})}{c^2}$$

that it is also constant. The final requirement on F comes from the equation (5.6) on a_{11}, namely that

$$(iii) \quad -2F_u F_p c = \frac{2}{c^2} a_{01}(a_{01}^2 - a_{00}c) = 0$$

We notice that a_{11} cannot be zero. If $a_{11} = 0$, the immediately $a_{01} = a_{00} = 0$. There are thus two possibilities for (iii) to be true:

(A) $a_{01} = 0$ which corresponds to $F_p = 0$
(B) $a_{00} = \frac{a_{01}^2}{c}$ which corresponds to $F_u = 0$

This can be formulated in the following way.

Theorem 5.4 *There are only two types of equations that possess a constant-coefficient g in the Cartan basis (up to multiplication by constants on g). They are*

(A) Equation $\quad y'' + \alpha y = f(x) \quad$ with $\quad g = \alpha\omega_0^2 + \omega_1^2$

(B) Equation $\quad y'' + \alpha y' = f(x) \quad$ with $\quad g = \alpha^2\omega_0^2 + 2\alpha\omega_0\omega_1 + \omega_1^2$

where $f(x)$ is arbitrary and $\alpha \in \mathbb{R}$.

5.4 Some equations with known co-/symmetries and g

In this section we are going to investigate equations of the type

$$y'' = y' + f(y)$$

where the function $f(y)$ is non-linear. That is, our $F(x, u, p) = p + f(u)$.
In the article [DL91] the problem of finding p-linear generating functions of symmetries is treated in full. All equations equipped with a two dimensional Lie-algebra of such symmetries are classified, although with a minor misprint.

The quest is to find solutions on the form $\phi = \alpha(x, u)p + \beta(x, u)$ to the Lie-equation

$$L\phi = 0$$

and thus bescrive $\mathcal{D}(\mathcal{C}_{\mathcal{E}})$ for this F.
But now we have also the problem of describing $\Omega^l(\mathcal{C}_{\mathcal{E}})$ and finding element(s) of $S^2(\mathcal{C}_{\mathcal{E}})$. We make the following observation about F.

$F_p = 1$, which by **Theorem 5.2** implies that

$$L(\phi) = 0 \quad \Leftrightarrow \quad L^*(e^{-x}\phi) = 0$$

Thus, solutions to the L-equation produces solutions to the L^*-equation and vice versa. Two independent solutions of the L^*-equation produce two independent cosymmetries , which in turn will give us an element of $S^2(\mathcal{C}_{\mathcal{E}})$.

Theorem 5.5 *Non-linear equations on the form $y'' = y' + f(y)$ that possess a two dimensional Lie-algebra of point-symmetries can be divided into classes* (1) *and* (2).

(1) $\quad y'' = y' + ae^{by} - \frac{2}{b} \qquad$ with $\ a, b \in \mathbb{R},\ a, b \neq 0$

(2) $\quad y'' = y' + a(y+b)^c - \frac{(2c+2)}{(c+3)^2}(y+b)$

with $\ a, b, c \in \mathbb{R}\ $ and $\ a \neq 0,\ c \neq 0, 1, -3$

These equations are equipped with the following structure, as listed below.

Type (1)

(i) Solutions of $L\phi = 0$:
$\phi_1 = p$ and $\phi_2 = e^{-x}(p - \frac{2}{b})$

Corresponding symmetries:
$X_1 = p\partial_u + (p + ae^{bu} - \frac{2}{b})\partial_p$ and
$X_2 = e^{-x}\left[(p - \frac{2}{b})\partial_u + ae^{bu}\partial_p\right]$

(ii) Solutions of $L^*\psi = 0$:
$\psi_1 = e^{-x}p$ and $\psi_2 = e^{-2x}(p - \frac{2}{b})$

Corresponding cosymmetries :
$\theta_1 = e^{-x}\left[(-p - ae^{bu} + \frac{2}{b})\omega_0 + p\omega_1\right]$ and
$\theta_2 = e^{-2x}\left[(-ae^{bu})\omega_0 + (p - \frac{2}{b})\omega_1\right]$

(iii) Metric structure :
$g = \theta_1^2 + \theta_2^2$ which corresponds to $a_{11} = \psi_1^2 + \psi_2^2$.

$$g = e^{-2x}\left[a^2(1 + e^{-2x})e^{2bu} + (p - \frac{2}{b})^2 + 2(p - \frac{2}{b})ae^{bu}\right]\omega_0^2$$

$$-2e^{-2x}\left[p(p - \frac{2}{b} + a) + a(p - \frac{2}{b})e^{bu-2x}\right]\omega_0{\cdot}\omega_1 + e^{-2x}\left[p^2 + e^{-2x}(p - \frac{2}{b})^2\right]\omega_1^2$$

Type (2)

(i) Solutions of $L\phi = 0$:
$\phi_1 = p$ and $\phi_2 = e^{kx}(p - \frac{k+1}{2}(u + b))$, where $k = \frac{1-c}{3+c}$

Corresponding symmetries:
$X_1 = p\,\partial_u + \left[p + a(u + b)^c - \frac{(2c+2)}{(c+3)^2}(u + b)\right]\partial_p$ and
$X_2 = e^{kx}\left[(p - \frac{k+1}{2}(u + b))\partial_u + (\frac{k+1}{2}p + a(u + b)^c - \frac{4}{(c+3)^2}(u + b))\partial_p\right]$

(ii) Solutions of $L^*\psi = 0$:
$\psi_1 = e^{-x}p$ and $\psi_2 = e^{(k-1)x}(p - \frac{k+1}{2}(u + b))$

Corresponding cosymmetries :

$\theta_1 = e^{-x}\,\left[(-p - a(u+b)^c + \frac{(2c+2)}{(c+3)^2}(u+b))\,\omega_0 \;+\; p\,\omega_1\right]$ and
$\theta_2 = e^{(k-1)x}\,\left[(-\frac{k+1}{2}p - a(u+b)^c - \frac{2c+2}{(c+3)^2}(u+b))\,\omega_0 \;+\; (p - \frac{k+1}{2}(u+b))\,\omega_1\right]$

(iii) Metric structure :
$g = \theta_1^2 + \theta_2^2$ which corresponds to $a_{11} = \psi_1^2 + \psi_2^2$.

$$g = e^{-2x}\left[\left(-p - a(u+b)^c + \frac{(2c+2)}{(c+3)^2}(u+b)\right)^2 + \right.$$

$$\left. +\; e^{2kx}\left(-\frac{(k+1)}{2}p - a(u+b)^c - \frac{(2c+2)}{(c+3)^2}(u+b) + \frac{(k+1)}{2}\right)^2\right]\omega_0^2 +$$

$$2e^{-2x}\left[p^2\left(-1 - \frac{(k+1)}{2}e^{2kx}\right) + a(u+b)^c p(-1 - e^{2kx}) \;+\right.$$

$$\left. e^{2kx}\left(\frac{(k^2+4k+3)}{4}p + \frac{(k+1)}{2}a(u+b)^c + \frac{(4c+4)}{(c+3)^3}(u+b) - \frac{(k+1)^2}{4}\right)\right]\omega_0{\cdot}\omega_1$$

$$+\,e^{-2x}\left[p^2 + e^{2kx}\left(p - \frac{(k+1)}{2}(u+b)\right)^2\right]\omega_1^2$$

5.5 Harmonic oscillator

We shall investigate the class of equations on the form

$$y'' + cy = 0 \tag{5.11}$$

where $c \in \mathbb{R}$. Whenever $c = a^2 > 0$ we have the famous **harmonic oscillator** equation.

Our first observation is that (5.11) is of type (A) in Theorem 5.4, and thus has a metric structure given by $g = c\omega_0^2 + \omega_1^2$. This g is positive for $c > 0$.

Second, we find the following if we search for p-linear solutions to the Lie-equation. The functions

$$\phi_1 = u \quad \text{and} \quad \phi_2 = p \tag{5.12}$$

solve $L\phi = 0$. They generate the following symmetries

$$X_1 = u\,\partial_u + p\,\partial_p \ , \qquad X_2 = p\,\partial_u - cu\,\partial_p \tag{5.13}$$

We note that $F = -cu \quad \Rightarrow \quad F_p = 0$ which by Theorem 5.2 implies that solutions of the L^*- equation must be scalar multiples of the solutions of the L-equation. Hence the functions

$$\psi_1 = u \quad \text{ and } \quad \psi_2 = p \tag{5.14}$$

solve the equation $L^*\psi = 0$, and provides us with the cosymmetries

$$\theta_1 = -p\,\omega_0 + u\,\omega_1 \ , \qquad \theta_2 = cu\,\omega_0 + p\,\omega_1 \tag{5.15}$$

The pairing

$$\langle X_1, \theta_2 \rangle = -\,\langle X_2, \theta_1 \rangle = cu^2 + p^2$$

is a 1st- integral of our equation. What is interesting, is that we get a new $g \in S^2(\mathcal{C}_{\mathcal{E}})$ from our cosymmetries by taking

$$g \ = \ \theta_1^2 + \theta_2^2 \ = \ [p^2 + c^2u^2]\,\omega_0^2 + 2\,[up(1-c)]\,\omega_0\,\omega_1 + [u^2 + p^2]\,\omega_1^2$$

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